Final Exam on Math 114 (Set Theory)
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Notes: The questions are not difficult as the number of points attached to them shows. Please read the text, understand the concepts and then read the questions attentively. Explain yourself clearly with short and correct English sentences. At most one idea per sentence! Use punctuation marks when necessary. Use paragraphs. Make sure that you state correctly what you know and what you are looking for. Draw pictures of sets, it will be helpful. Do not use logical symbols such as ⇒, &, ∀.

A ***topological space*** is a set X together with some specific subsets of X called ***open*** that satisfy the following properties:

T1. $\varnothing$ and X are open subsets.

T2. The intersection of two open subsets is open.

T3. The union of any set of open subsets is open.

Assuming that the open subsets of X are known, by abuse of language we say that X is a topological space. But very often we will consider several topologies on the same set X.

If τ is the set of open subsets of a topological space X, it is sometimes convenient to define a topological space as the pair (X, τ). Then the elements of τ (i.e. the open subsets of the topological space X) satisfy the following properties:

T1′. $\varnothing \in \tau$ and $X \in \tau$.

T2′. If $U, V \in \tau$, then $U \cap V \in \tau$.

T3′. If $\sigma \subseteq \tau$, then $\cup\sigma \in \tau$.

Note that $\tau \subseteq \wp(X)$, i.e. $\tau \in \wp(\wp(X))$. Sometimes we also say that a subset τ of $\wp(X)$ that satisfies T1′, T2′ and T3′ is a ***topology*** on X.

If τ_1 and τ_2 are two topologies on the same set X and if $\tau_1 \subseteq \tau_2$ then we say that τ_1 is ***coarser*** than τ_2 or that τ_2 is ***richer*** than τ_1

1. Let X be any set.

1a. Show that there is a unique topology (called ***coarsest topology***) on X which is coarser than any topology on X. (2 pts.)

1b. Show that there is a unique topology (called ***discrete topology***) on X which is richer than any topology on X. (2 pts.)

1c. Show that if Σ be a set of topologies on X then $\cap\Sigma$ is a topology on X. (3 pts.)

1d. Given a set $\tau \subseteq \wp(X)$ of subsets of X, show that there is a unique topology on X which contains τ (as a subset of course, hence the elements of τ will be open subsets in the topology) and which is coarser than any topology that contains τ. (5 pts.)

We say that this topology (of 1d) is ***generated*** by τ and we denote it by $\langle\tau\rangle$.

1e. If $\tau = \{A, B, C\} \subseteq \wp(X)$, find $\langle\tau\rangle$. (2 pts.)

1f. Find $\langle\varnothing\rangle$. (2 pts.)

1g. Let $\tau \subseteq \wp(X)$ be such that for any $A, B \in \tau$, the intersection $A \cap B$ is a union of sets from τ. Show that $\langle\tau\rangle = \{\cup\sigma : \sigma \subseteq \tau\}$. (4 pts.)

1h. Show that if $\tau_1 \subseteq \tau_2$ then $\langle\tau_1\rangle \subseteq \langle\tau_2\rangle$. (2 pts.)

1i. For $\tau \subseteq \wp(X)$ let $\tau_{\text{int}} = \{A_1 \cap ... \cap A_n : n \in \mathbb{N} \text{ and } A_i \in \tau \text{ for all } i = 1, ..., n\}$. Show that τ_{int} satisfies the premisses of 1g and that $\langle\tau\rangle = \langle\tau_{\text{int}}\rangle$. (4 pts.)

2. Let Y be a topological space and X a subset of Y. Call a subset U of X open in X if it is the intersection of an open subset of Y with X; i.e. a subset $U \subseteq X$ is called open in X if $U = V \cap X$ for some open subset V of Y. Show that this defines a topology on X. This topology on X is called **the restricted topology** (from the topology of Y). (4 pts.)

3. Let X and Y be two topological spaces. Call a subset of $X \times Y$ open if it is a union of sets of the form $U \times V$ where U and V are open in X and Y respectively. Show that this defines a topology on $X \times Y$. (3 pts.) This topology is called ***product topology on*** $X \times Y$.

4. A function $f : X \to Y$ from a topological space X into another topological space Y is called ***continuous*** if for any open subset V of Y, $f^{-1}(V)$ is an open subset of X. Thus the continuity of a map $f : X \to Y$ depends on the topologies on X and Y.

4a. Let X, Y, Z be three topological spaces. Show that if $f : X \to Y$ and $g : Y \to Z$ are continuous then so is the composite $g \circ f : X \to Z$. (2 pts.)

4b. Show that a constant map between two topological spaces is continuous. (3 pts.)

4c. Let X be a topological space. Show that $\mathrm{Id}_X : X \to X$ is continuous. (Here the topology on the departure and the arrival sets are supposed to be the same.) (1 pt.)

4d. Let X be a set with at least two elements. Show that $\mathrm{Id}_X : X \to X$ is not continuous if on the departure set we take the coarsest topology and on the arrival set the discrete topology. (2 pts.)

4e. Let Y be a topological space and $X \subseteq Y$. Consider X as a topological space with the restricted topology from the topology of Y (see #2). Show that the inclusion map $i : X \to Y$ given by $i(x) = x$ is continuous. (4 pts.)

4f. Let X and Y be two topological space. Consider $X \times Y$ with the product topology (see #3). Show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous. (3 pts.)

4g. Show that if Y has the coarsest topology (see 1a) then any function f from any topological space into Y is continuous. (2 pts.)

4h. Show that if X has the discrete topology (see 1b) then any function f from X into any topological space Y is continuous. (2 pts.)

4i. Let X be a set, Y a topological space and $f : X \to Y$ a map. Let τ and τ_1 be two topologies on X and suppose that τ_1 is richer than τ. Show that if f is continuous when X is considered as a topological space with τ then f is still continuous when X is considered as a topological space with τ_1. (2 pts.)

4j. Let X be a set, Y a topological space and $f : X \to Y$ a map. Let Σ be the set of all topologies on X for which f is continuous. Show that f is continuous when X is considered as a topological space with $\cap\Sigma$. (2 pts.) Show that $\cap\Sigma = \langle f^{-1}(V) : V \text{ open subset of } Y\rangle$. (6 pts.) Conclude that $\cap\Sigma$ is the smallest topology on X that makes f continuous. (3 pts.)

4k. Let Y be a topological space and $X \subseteq Y$ be a subset. Let $i : X \to Y$ be the inclusion map given by $i(x) = x$. Let τ be the smallest topology on X that makes i continuous. Show that τ is the restricted topology on X (from the topology on Y). (8 pts.)

4l. Let X be a set, Y a topological space and $\mathcal{F}$ a set of functions from X into Y. Show that there is a smallest topology on X that makes all the maps of $\mathcal{F}$ continuous. (4 pts.)

4m. Let X and Y be two topological spaces. Show that the smallest topology on $X \times Y$ that makes the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ continuous is the product topology on $X \times Y$ (see #3). (10 pts.)

4n. Let X be a topological space and I a set. Consider the set $\Pi_I X$ of all functions from I into X. Given $f \in \Pi_I X$ and $i \in I$, set $\pi_i(f) = f(i)$. Then π_i is a function from $\Pi_I X$ into X (called the i^{th} projection map). Find the smallest topology on $\Pi_I X$ for which all the projection maps are continuous. (10 pts.)

5. Let X be a topological space and K a subset of X. A family $(U_i)_{i\in I}$ of subsets of X is called a ***cover*** of K if $K \subseteq \cup_{i\in I} U_i$. Such a cover is called an ***open cover*** of K if each U_i is open. A ***subcover*** of $(U_i)_{i\in I}$ is cover of the form $(U_j)_{j\in J}$ for some subset J of I. A subset K of a topological space is called ***compact*** if every open cover of K has a finite subcover.

5a. Show that the union of finitely many compact subsets is compact. (4 pts.)

5b. Show that a finite subset is compact. (4 pts.)

5c. Show that if $f : X \to Y$ is a continuous map between two topological spaces then the image of any compact subset of X is a compact subset of Y. (6 pts.) Is it true that the preimage of a compact subset of Y is always a compact subset of X? (3 pts.)

5d. A subset C of a topological space X is called ***closed*** if its complement $X \setminus C$ is open. Show that a closed subset of a compact set is compact. (6 pts.)

5e. A topological space is called **compact** if it is a compact subset of itself. Let X be a topological space and Y a subset of X. Show that Y is a compact subset of X if and only if it is a compact topological space with the restricted topology (see #2). (6 pts.)

5f. Let X and Y be two topological spaces. Show that $X \times Y$ is a compact topological space with the product topology (see #3) if and only if both X and Y are compact spaces. (10 pts.)