Math 111 Midterm

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Warning: Write legibly, clearly, in proper English and with short sentences. Use proper punctuation. Make paragraphs; one idea per paragraph. Do not use symbols such as $\forall$, $\exists$, $\Rightarrow$. For each use of such a symbol, your grade will be deduced by 1/100. Do not use unnecessary mathematical symbols.

Do not try to chase for points by trying to do all the problems, because partial credit will be given only rarely. In fact it is not the correct answer that will get points, but the idea behind it.

The number of points of each problem is intended to indicate its difficulty.

Notation: If X is a set, $\wp(X)$ denotes the set of subsets of X and Id_X denotes the identity map on X, i.e. $\mathrm{Id}_X(x) = x$ for all $x \in X$. The symbol $f \circ g$ denotes the composition of the functions f and g whenever it makes sense; thus $(f \circ g)(x) = f(g(x))$. The symbol f^2 denotes the function $f \circ f$ whenever it makes sense.

1. Write all the elements of $\wp(\wp(\{1,2\}))$. (4 pts.)

2. Do the following sets have a maximal and a minimal element (for the inclusion)? (6 pts.)

a) $\{x \in \wp(\mathbb{N}): 1 \in x, 2 \notin x\}$

b) $\{x \in \wp(\mathbb{N}): \exists n \in \mathbb{N}\ \forall m > n\ (m \notin x)\}$

c) $\{x \in \wp(\wp(\mathbb{N})): a\mathbb{N} \in x$ for all $a \in \mathbb{N}\}$

3. Find all the bijections f of the set $4 := \{0, 1, 2, 3\}$ for which $f^3 = \mathrm{Id}_4$. (3 pts.)

4. Find all the bijections $f: \mathbb{N} \to \mathbb{N}$ for which

$$x < y \Rightarrow f(x) < f(y)$$

for all $x, y \in \mathbb{N}$. (5 pts.) Do the same with $\mathbb{Z}$ instead of $\mathbb{N}$. (5 pts.)

5. Let $f: \mathbb{N} \to \mathbb{Z}$ be given by $f(x) = x - x^2$ and $g: \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x/(1+x^2)$. Compute $(g \circ f)(3)$. For what values of x, does $(g \circ f)(x) \geq 0$? (4 pts.)

6. Let X be a set and $f: X \to X$ be a function such that f^2 is a bijection. Show that f is a bijection. (3 pts.)

7. Give an example of a nonconstant function $f: \mathbb{N} \to \mathbb{N}$ such that $f \neq \mathrm{Id}_{\mathbb{N}}$ and $f^2 = f$. (3 pts.)

8. Find a bijection $f: \mathbb{N} \to \mathbb{N}$ such that for all $k > 0$ and n, $f^k(n) \neq n$. (10 pts.)

9. Show that for any nonzero natural number n,

$$\frac{1}{1}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{5} + \frac{1}{5}\times\frac{1}{7} + \ldots + \frac{1}{2n-1}\times\frac{1}{2n+1} = \frac{n}{2n+1}$$

(5 pts.)

10. Let X be a set. Let 2^X denote the set of functions fron X into $2 = \{0, 1\}$. Show that there is a bijection between 2^X and $\wp(X)$. (6 pts.)

11. Let X and Y be two set. Let X^Y denote the set of functions fron Y into X. Show that there is a bijection between $(X^Y)^Z$ and $X^{Y\times Z}$. (10 pts.) Conclude that there is a bijection between 2^ω and $(2^\omega)^\omega$. (10 pts.) Conclude that there is a one-to-one map from 3^ω into 2^ω. (6 pts.)

12. Let X be a nonempty subset of $\mathbb{Z}$ such that for all $x, y \in X$, $x - y \in X$. Show that $X = n\mathbb{Z}$ for some unique $n \in \mathbb{N}$. (10 pts.)

13. For $(x, y), (z, t) \in \mathbb{Z} \times \mathbb{Z}$ define $(x, y) - (z, t) = (x - z, y - t)$. Let X be a nonempty subset of $\mathbb{Z} \times \mathbb{Z}$ such that for all $\alpha, \beta \in X$, $\alpha - \beta \in X$. Show that $X = \alpha\mathbb{Z} + \beta\mathbb{Z}$ for some $\alpha, \beta \in X$. (Here $\alpha\mathbb{Z} + \beta\mathbb{Z}$ denotes the set $\{n\alpha + m\beta : n, m \in \mathbb{Z}\}$, 30 pts.)