Algebra Math 212
Quiz

1. Let $G$ be a group and $K$ a field. In this and the next exercise, it is advised to write $G$ multiplicatively. Consider the formal elements of the form

$$\sum_{g \in G} \lambda_g g$$

where $\lambda_g \in K$ and only finitely many of them are nonzero. Let $K[G]$ be the set of such elements.

1a. Find the elements of $F_2[\mathbb{Z}/3\mathbb{Z}]$. (2 pts.)

Define $+$, $\times$ and scalar multiplication formally on $K[G]$ as follows:

$$\left( \sum_{g \in G} \lambda_g g \right) + \left( \sum_{g \in G} \mu_g g \right) = \sum_{g \in G} (\lambda_g + \mu_g) g$$

$$\left( \sum_{g \in G} \lambda_g g \right) \times \left( \sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \left( \sum_{hk=g} \mu_h \lambda_k \right) g$$

Then $K[G]$ becomes a (not necessarily commutative) ring with 1 and also a $K$-vector space satisfying $\lambda(ab) = (\lambda a)b = a(\lambda b)$ (such a structure is called an algebra or a $K$-algebra, e.g. $\text{End}_K(V)$ is a $K$-algebra).

1b. Show that $G \leq K[G]^*$. (2 pts.)

1c. Find the invertible and the nilpotent elements and the idempotents of $F_2[\mathbb{Z}/3\mathbb{Z}]$. (4 pts.)

1d. If $G$ is finite what is $\left( \sum_{g \in G} \lambda_g g \right)^2$? (4 pts.)

1e. Show that if $G$ has torsion elements, then $K[G]$ has zero-divisors. (2 pts.)

1f. Show that $K[\mathbb{Z}]$ has no zero-divisors. (4 pts.)

1g. Show that the set of elements of the form $\sum_{g \in G} \lambda_g g$ where $\sum_{g \in G} \lambda_g = 0$ forms an ideal of $K[G]$. (3 pts.)

1h. Let $G$ be a group, $K$ a field and $\varphi : G \to \text{GL}(V) \subseteq \text{End}_K(V)$ a group homomorphism. Show that $\varphi$ extends uniquely to a $K$-algebra homomorphism $\varphi : K[G] \to \text{End}_K(V)$. (3 pts.)

1i. Notation as above. Show that defining $av$ as $\varphi(a)(v)$ for $a \in K[G]$ and $v \in G$, $V$ becomes a $K[G]$-module via $\varphi$. (1 pt.)

2. The purpose of this exercise is to prove Maschke’s Theorem that states the following: Let $G$ be a finite group, $K$ a field whose characteristic does not divide $|G|$ and $V$ a $K[G]$-module. Then $V$ is completely reducible, i.e. any submodule of $V$ has a complement in $V$.

2a. Show that a vector space endomorphism $u$ of $V$ is a $K[G]$-module endomorphism iff $u(gv) = gu(v)$ for all $g \in G$ and $v \in V$. (3 pts.)

2b. Let $W$ be a $K[G]$-submodule of $V$. Let $U$ be a complement of $W$ in $V$ (as a vector space over $K$). Thus $V = W \oplus U$. Let $\pi$ be the projection of $V$ onto $W$ according to this decomposition. Let $u : V \to V$ be defined by $u(v) = \sum_{g \in G} g \pi(g^{-1}v)$. Show that $u(V) \leq W$, that $u$ is a $K[G]$-module homomorphism, that in case $G$ is finite $u|_W = |G| \text{Id}_W$ and that $u \circ u = |G| u$. (10 pts.)
2c. Assume now that \( G \) is finite and that char(\( K \)) does not divide \(|G|\). Let \( v = \frac{1}{|G|} u \). Show that \( V = W \oplus \text{Ker}(v) \). (Now \( \text{Ker}(v) \) is a \( K[G] \)-module.) (8 pts.)

2d. Show that if further \( \dim_K(V) < \infty \) then \( V \) is a direct sum of irreducible modules. (5 pts.)