CourseWork Questions

2a. Let $(E_n)_{n \in \mathbf{N}}$ be a sequence of disjoint sets and for each n, $m \in N$ such that $n \leq m$, let f_{mn} be a function from E_n into E_m. Suppose that the functions f_{nm} satisfy the following two properties:

a) $f_{nn} = \mathrm{Id}_{E_n}$ for all n.

b) $f_{pn} = f_{pm} \circ f_{mn}$ for all $p \geq m \geq n$.

Let E be the union of the sets E_n. For x, $y \in E$, define: $x \approx y$ iff $x \in E_n$, $y \in E_m$ and there is a p greater than n and m such that $f_{pn}(x) = f_{pm}(y)$.

Show that $\approx$ is an equivalence relation on E.

2b. Show that $x \approx y$ iff for $s = \max(n, m)$, the sequences $(f_{pn}(x))p> s$ and $(f_{pn}(y))p> s$ coincide for large enough p.

2c. Suppose now that each E_n is a group and that the functions *fnm* are group homomorphisms. For $x \in E$, let $[x]$ denote the equivalence class of x. (So $[x] \in E/\approx$).

For x, $y \in E$, define $[x]\ [y] = [f_{pn}(x)f_{pm}(y)]$ where n, m, p are such that $x \in E_n$, $y \in E_m$, $p > n$ and $p > m$.

Show that this is a well-defined product and that it turns $E/\approx$ into a group.

3a. Let p be a prime and let $E_n = \mathbb{Z}/p^n\mathbb{Z}$. For $m \geq n$, define f_{mn}: $E_n \to E_m$ by $f_{mn}(x) = p^{m-n}x$. Show that the functions fmn are well-defined and that they satisfy the hypothesis of Question 2a.