CourseWork Questions

2a. Let \((E_n)_{n \in \mathbb{N}}\) be a sequence of disjoint sets and for each \(n, m \in \mathbb{N}\) such that \(n \leq m\), let \(f_{mn}\) be a function from \(E_n\) into \(E_m\). Suppose that the functions \(f_{mn}\) satisfy the following two properties:

   a) \(f_{nn} = I_{E_n}\) for all \(n\).

   b) \(f_{pm} = f_{pm} \circ f_{mn}\) for all \(p \geq m \geq n\).

   Let \(E\) be the union of the sets \(E_n\). For \(x, y \in E\), define: \(x \approx y\) iff \(x \in E_n, y \in E_m\) and there is a \(p\) greater than \(n\) and \(m\) such that \(f_{pn}(x) = f_{pm}(y)\).

   Show that \(\approx\) is an equivalence relation on \(E\).

2b. Show that \(x \approx y\) iff for \(s = \max(n, m)\), the sequences \((f_{pn}(x))_{p > s}\) and \((f_{pn}(y))_{p > s}\) coincide for large enough \(p\).

2c. Suppose now that each \(E_n\) is a group and that the functions \(f_{nm}\) are group homomorphisms. For \(x \in E\), let \([x]\) denote the equivalence class of \(x\). (So \([x]\) \(\in E/\approx\)).

   For \(x, y \in E\), define \([x]\) \([y] = [f_{pn}(x)f_{pm}(y)]\) where \(n, m, p\) are such that \(x \in E_n, y \in E_m, p > n\) and \(p > m\).

   Show that this is a well-defined product and that it turns \(E/\approx\) into a group.

3a. Let \(p\) be a prime and let \(E_n = \mathbb{Z}/p^n\mathbb{Z}\). For \(m \geq n\), define \(f_{mn}: E_n \rightarrow E_m\) by \(f_{mn}(x) = p^{m-n}x\). Show that the functions \(f_{mn}\) are well-defined and that they satisfy the hypothesis of Question 2a.