Math 112 Final Exam
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1. Let B be a set. Let $A \subseteq B$. Assume that A is well-ordered say by $<$. Show that the well-order $<$ on A can be extended to a well-order on B in such a way that A is an initial segment of B. (10 pts.)

Proof: Let Z be the set of well-ordered sets $(X, <)$ such that

a) $A \subseteq X \subseteq B$,
b) $(X, <)$ extends the order $<$ of A,
c) A is an initial segment of X.

Order Z as follows: $(X, <_X) \leq (Y, <_Y)$ if

a) $X \subseteq Y$,
b) $<_Y$ extends $<_X$
c) X is an initial segment of Y.

Then Z is nonempty and is an inductive set: If $(X_i, <_i)_{i\in I}$ is a chain from Z then $\cup_{i\in I} X_i$ can be well-ordered naturally extending the order of each X_i and as such $X_i \leq \cup_{i\in I} X_i$ for each $i \in I$.

It follows from Zorn's Lemma that Z has a maximal element, say $(C, <)$. If $C \subset B$, then take any $b \in B \setminus C$ and extend the order of C to the set $C \cup \{b\}$ by putting b to the very end of C. Then $(C, <) < (C \cup \{b\}, <)$ and this contradicts the maximality of $(C, <)$.

A **cardinal number** is an ordinal number κ such that there is no bijection between κ and any ordinal $\alpha < \kappa$.

2. Show that each natural number is a cardinal number. (5 pts.)

Proof: We will show that, for any $n \in \omega$, there is no one-to-one map from n into any $m < n$. This will prove that each $n \in \omega$ is a cardinal number. Clearly 0 is a cardinal number. Assuming $n \in \omega$ is a cardinal number we show that $S(n)$ is a cardinal number. Assume there is an $m \in S(n) = n \cup \{n\}$ and a one to one map $f : S(n) \to m$. Set $f(n) = i \in m$. Then f induces a one to one map $g : n \to m \setminus \{i\}$. Since $i \in m$, $m \neq 0$; so there is a k such that $m = S(k)$. Define $h : m \setminus \{i\} \to k$ by $h(j) = j$ if $j < i$ and $h(j) = j - 1$ if $j > i$. Then h is a bijection from $m \setminus \{i\}$ into k. Now $h \circ g : n \to k$ is a one to one map. Since $k < S(k) = m \leq n$, this contradicts the inductive assumption.

3. Show that ω is a cardinal number. (4 pts.)

Proof: We need to show that there is no one to one map from ω into a natural number $n \in \omega$. We proceed by induction. If $n = 0$ this is clear. Let $f : \omega \to S(n)$ be a one to one map. Assume $f(0) = i \in S(n)$. Then $f \circ S : \omega \to S(n) \setminus \{i\}$ is a one to one map. As above, we can find a bijection g between $S(n) \setminus \{i\}$ and n. Thus $g \circ f \circ S$ is a one to one map from ω into n and this contradicts the inductive assumption.

4. Find an ordinal which is not a cardinal. (2 pts.)

Proof: $S(\omega)$ is not a cardinal because the map $f : S(\omega) \to \omega$ defined by $f(\omega) = 0$ and $f(n) = S(n)$ for all $n \in \omega$ is a bijection.

5. Show that if X is a set then there is a unique cardinal number κ such that there is a bijection between X and κ. This κ is called the cardinality of X and is denoted by $|X|$. (5 pts.)

Proof: By taking $A = \varnothing$ in Q1, we know that X can be well-ordered. Thus there is a bijection between X and an ordinal, say α. Let $U = \{\beta \in S(\alpha) :$ there is a bijection from X onto $\beta\}$. Then U is a nonempty subset of $S(\alpha)$ because $\alpha \in U$. Hence U has a least element, say β. Thus there is a bijection between X and β and there is no bijection between X and any ordinal $\gamma < \beta$. Hence β must be a cardinal number.

6. Show that if X is infinite (meaning $|X| \geq \omega$) and $x \in X$ then $|X \setminus \{x\}| = |X|$. (8 pts.)

Proof: By Q1 we can well order X so that x is the least element. Thus we can assume that $X = \kappa$ is a cardinal number and that $x = 0$. We have to show that there is a bijection between κ and $\kappa \setminus \{0\}$. Since κ is infinite, $\omega \subseteq \kappa$ and it is enough to show that there is a bijection between ω and $\omega \setminus \{0\}$. This is easy: S is such a bijection.

7. Show that if α is a cardinal number $\geq \omega$ then α is a limit ordinal. (5 pts.)

Proof: We have to show that if α is an infinite ordinal then $S(\alpha)$ is not a cardinal. We will show that there is a bijection between $S(\alpha)$ and α, proving that $S(\alpha)$ cannot be an ordinal. Setting $X = S(\alpha)$ in Q7, we obtain the result.

8. Show that for each cardinal number κ there is a unique cardinal number λ such that $\kappa < \lambda$ and that there is no cardinal number α such that $\kappa < \alpha < \lambda$. We denote this cardinal number by κ^+. (6 pts.)

Proof: Let $|\wp(\kappa)| = \lambda$. If $\lambda \leq \kappa$ then there would be a one to one map f from $\wp(\kappa)$ into κ. Define $g : \kappa \to \wp(\kappa)$ by $g(\alpha) = f^{-1}(\alpha)$ if $\alpha \in f(\wp(\kappa))$ and let $g(\alpha) = \varnothing$ if $\alpha \notin f(\wp(\kappa))$. Then g is onto. But as we know there is no such map.

9. Let α and β be two cardinal numbers. Find two disjoint sets A and B such that $|A| = \alpha$ and $|B| = \beta$. (2 pts.)

Proof: Let $A = \alpha \times \{0\}$ and $B = \beta \times \{1\}$.

10. If α, β, A and B are as above, we define,

$$\alpha + \beta = |A \cup B|$$
$$\alpha\beta = |A \times B|$$
$$\alpha^\beta = |A^B|.$$

(Here A^B is the set of functions from B into A). What do you need to show for this to be a valid definition? (2 pts.) Do we really need A and B to be disjoint in all three definitions? (3 pts.)

11. Show that $\kappa < 2^\kappa$. (4 pts.)

12. For any cardinal number κ, find $\kappa + 0$, 0κ, κ^0, 0^κ, 1κ, κ^1, 1^κ (3 pts.)

13. Show that for any cardinal numbers α, β, γ,

$$\alpha + \beta = \beta + \alpha$$
$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$
$$\alpha\beta = \beta\alpha$$
$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$
$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$
$$\alpha^\beta\alpha^\gamma = \alpha^{\beta+\gamma}$$
$$(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$$
$$\alpha^\gamma\beta^\gamma = (\alpha\beta)^\gamma$$

(16 pts.)

14. Show that if at least one of α and β is an infinite cardinal number and if thety are both nonzero then $\alpha + \beta = \max\{\alpha, \beta\}$ and $\alpha\beta = \max\{\alpha, \beta\}$. (5 pts.)

15. Show that if $\alpha \leq \beta$ and $\gamma \leq \delta$ are cardinal numbers, then

$$\alpha + \gamma \leq \beta + \delta$$
$$\alpha\gamma \leq \beta\delta$$
$$\alpha^{\gamma} \leq \beta^{\delta} \text{ unless } \alpha = \gamma = \beta = 0 < \delta$$

(9 pts.)

14. Let $(\alpha_i)_{i\in I}$ be a family of cardinal numbers. Show that there is a family $(A_i)_i$ of sets such that $|A_i| = \alpha_i$ and $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$. (2 pts.)

15. Let $(\alpha_i)_{i\in I}$ and $(A_i)_i$ be as above. Define

$$\Sigma_{i\in I}\, \alpha_i = |\cup_{i\in I} A_i|$$

and

$$\Pi_{i\in I}\, \alpha_i = |\Pi_{i\in I} A_i|.$$

Do we need the sets A_i to be pairwise disjoint in both definitions? (2 pts.)

16. Let $(\alpha_i)_{i\in I}$ be a family of cardinal numbers. Assume that for some cardinal number α, $\alpha_i = \alpha$ for all i. Show that $\Sigma_{i\in I}\, \alpha_i = \alpha|I|$ and $\Pi_{i\in I}\, \alpha_i = \alpha^{|I|}$. (6 pts.)

17. Let $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ be two families of cardinal numbers. Assume that $\alpha_i \leq \beta_i$ for all $i \in I$. Show that $\Sigma_{i\in I}\, \alpha_i \leq \Sigma_{i\in I}\, \beta_i$ and that $\Pi_{i\in I}\, \alpha_i \leq \Pi_{i\in I}\, \beta_i$. (6 pts.)

18. Let $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ be two families of cardinal numbers. If $\alpha_i < \beta_i$ for all i then $\Sigma_{i\in I}\, \alpha_i < \Pi_{i\in I}\, \beta_i$ (König's Theorem). (20 pts.)