Math 111 - Set Theory
Midterm

May 2007

Ali Nesin

Write correctly, with correct puctuation and make simple and full sentences. Do not use logical symbols such as $\forall$, $\exists$, $\Rightarrow$.

The first part is much easier than the second. I advise **strongly** not to attack the second part unless you are sure that you succeeded the first part.

I don't wish good luck since there will be no luck involved!

Let X be a nonempty set, which will be fixed throughout the exam. A **filter** on X is a set $\mathcal{F}$ of subsets of X (hence $\mathcal{F} \subseteq \wp(X)$) that satisfies the following three properties:

i) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

ii) If A and B are in $\Im$, then so is $A \cap B$.

iii) $\varnothing \notin \mathcal{F}$ and $X \in \mathcal{F}$.

Part I

1. Show that if $\varnothing \neq A \subseteq X$ then the set of subsets $\mathcal{F}(A)$ of X that contain A is a filter on X. Such a filter is called **principal**.

2. Show that if $\varnothing \neq A \subseteq B \subseteq X$ then $\mathcal{F}(B) \subseteq \mathcal{F}(A)$.

3. Show that for $A, B \subseteq X$, $\mathcal{F}(A) \cap \mathcal{F}(B) = \mathcal{F}(A \cup B)$.

4. Show that a filter that contains a finite subset of X is necessarily principal.

5. Suppose X is infinite. A subset A of X is called **cofinite** if its complement $X \setminus A$ is finite. Show that the set of cofinite subsets of X is a nonprincipal filter on X. This filter is called the **Fréchet filter** (on X).

6. Show that a nonprinciple filter necessarily contains the Fréchet filter.

7. Show that the intersection of any set of filters is a filter.

8. Let $(I, <)$ be an ordered set such that for each $i, j \in I$ there is a $k \in I$ such that $i \leq k$ and $j \leq k$. For each $i \in I$, let $\mathcal{F}_i$ be a filter on X. Suppose that for $i < j$, $\mathcal{F}_i \subseteq \mathcal{F}_j$ and that $\{\mathcal{F}_i : i \in I\}$ is a set. Show that $\cup_{i \in I} \mathcal{F}_i$ is a filter.

9. Let $\mathcal{B} \subseteq \wp(X)$ be contained in a filter. Show that there are no $A_1, ..., A_n \in \mathcal{B}$ such that $A_1 \cap ... \cap A_n = \varnothing$. A subset $\mathcal{B}$ of $\wp(X)$ satisfying this property is said to have **the finite intersection property**, **FIP** for short.

10. Conversely show that any $\mathcal{B} \subseteq \wp(X)$ that satisfies FIP is contained in a filter.

11. A filter is called **ultrafilter** if it is a maximal filter, i.e. if it is not a proper subset of another filter. For what subsets $A \subseteq X$, is the principal filter $\mathcal{F}(A)$ an ultrafilter?

12. Show that a filter $\Im$ on X is an ultrafilter if and only if for any $A \subseteq X$, either A or $X \setminus A$ is in $\mathcal{F}$.

Part II

13. Let $\mathcal{F}$ be a filter on a set X. Let Y be a set. Denote the set of functions from X into Y by Func(X, Y). On Func(X, Y) define the following binary relation:

$$f \equiv g \Leftrightarrow \{x \in X : f(x) = g(x)\} \in \mathcal{F}.$$

Show that this is an equivalence relation on the set Func(X, Y).

From now on we fix two sets X and Y, a filter $\mathcal{F}$ on X and we set $\equiv$ as above. We let Func(X, Y)/$\equiv$.

14. Let $\leq$ be a partial order on Y. On Func(X, Y) define the relation $\preccurlyeq$ by

$$f \preccurlyeq g \text{ if and only if } \{x \in X : f(x) \leq g(x)\} \in \mathcal{F}.$$

a) Is this a partial order on Func(X, Y)? If not what property of the partial orders fail to hold?

b) Show that if $f \preccurlyeq g$ and $f \equiv f_1$ and $g \equiv g_1$ then $f_1 \preccurlyeq g_1$.

c) Conclude from above that $\leq$ gives rise to a binary relation on Func(X, Y)/$\equiv$.

d) Suppose now that $\leq$ is a total order on Y and that $\Im$ is an ultrafilter on X. Conclude from above that the binary relation on Func(X, Y)/$\equiv$ defined is a total order.

e) Suppose now that $\leq$ is a well-order on Y and that $\Im$ is an ultrafilter on X. Is the binary relation on Func(X, Y)/$\equiv$ defined above a well-order?

15. Let $n \in \mathbb{N}$ be any natural number > 0. (To start with you may take $n = 1$). Let $f : Y^n \rightarrow Y$ be a function. We will define a function

$$f^* : \text{Func}(X, Y)^n \rightarrow \text{Func}(X, Y).$$

For $g_1, ..., g_n \in$ Func(X, Y), $f^*(g_1, ..., g_n)$ should be defined to be an element of Func(X, Y), i.e. should be a function from X into Y. To define such a function we must tell its value at an arbitrary element $x \in X$. This value is defined as follows:

$$f^*(g_1, ..., g_n)(x) = f(g_1(x), ..., g_n(x)),$$

which is really an element of Y. This defines f^*. Now the question: Show that if $g_1, ..., g_n, h_1, ..., h_n \in$ Func(X, Y) are such that $g_1 \equiv h_1, ..., g_n \equiv h_n$ then $f^*(g_1, ..., g_n) \equiv f^*(h_1, ..., h_n)$.

16. With the above question in mind, show why any function $f : Y^n \rightarrow Y$ gives rise to a function $[f]$: Func$(X, Y)^n \rightarrow$ Func(X, Y).