Set Theory (Math 111)
In Class Midterm on Rational Numbers

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1) Do not use symbols such as $\Rightarrow$, $\forall$. 2) Make full sentences. 3) Write legibly. 4) Use correct punctuation. 5) Explain your ideas clearly. 6) You may use all the elementary facts about the structure $\langle \mathbb{Z}, +, \times, \leq, 0, 1\rangle$ without proof.

Let $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. On X we define the binary relation:

$$(x, y) \equiv (z, t) \iff xt = yz.$$

1. Show that $\equiv$ is an equivalence relation on X.

   For $(x, y) \in X$, we let $\overline{(x, y)}$ denote the equivalence class of (x, y). We also set
   $$\mathbb{Q} = X/\equiv = \{\overline{(x, y)} : (x, y) \in X\}.$$

2. Show that for any $(x, y) \in X$ there is an $(x', y') \in X$ such that $\overline{(x, y)} = \overline{(x', y')}$ and $y' > 0$.

3. Let (x, y), (z, t), (x', y'), $(z', y') \in X$. Assume $\overline{(x, y)} = \overline{(x', y')}$ and $\overline{(z, t)} \equiv \overline{(z', t')}$. Show that

   i. $\overline{(xt \pm yz, yt)} = \overline{(x't' \pm y'z', y't')}$.

   ii. $\overline{(xz, yt)} = \overline{(x'z', y't')}$.

   iii. If $z \neq 0$, then $z' \neq 0$ and $\overline{(xt, yz)} = \overline{(x't', y'z')}$.

4. Explain why we are now entitled to define four operations, that we will call addition, substraction, multiplication and division (for the division we have to assume that $z \neq 0$), as follows:

$$\begin{aligned} \overline{(x, y)} \pm \overline{(z, t)} &= \overline{((xt \pm yz, yt)} \\ \overline{(x, y)} \times \overline{(z, t)} &= \overline{(xz, yt)} \\ \overline{(x, y)}/\overline{(z, t))} &= \overline{(xt, yz)} \end{aligned}$$

5. Find the zero element for the addition and the identity element for the multiplication. Show that each element of $\mathbb{Q}$ has an additive inverse and that each nonzero element has a multiplicative inverse. Prove the distributivity law.

6. Define a total order on $\mathbb{Q}$ and state the lemmas (without proving) that one needs to prove in order for your definition to make sense. For $\alpha \in \mathbb{Q}$ define $\alpha/2$ and prove that for all $\alpha < \beta$ in $\mathbb{Q}$, $\alpha < (\alpha + \beta)/2 < \beta$.