Math 111
Resit

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Important Note. Write either in English or in Turkish, but in any event make full sentences. Use proper punctuation. Do not use symbols such as $\Leftrightarrow$, $\Rightarrow$, $\exists$, $\forall$. For each use of these symbols I will take away 1 pt. Explain all your answers, but grades will be taken away for unnecessary text. Any unexplained answer will get 0 point, whether the answer is correct or not.

I. Transitive Relations. Let X be a set. A **binary relation** on X is just a subset of $X \times X$.

A binary relation R on X is called **transitive** if, for all x, y, $z \in X$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

i. Which of the following a transitive binary relation on any set X? Explain. (0 or 4 pts.)

i. $X \times X$.

ii. $\emptyset$.

iii. $\{(x, x) : x \in X\}$.

iv. $\{(x, y) \in X^2 : x \neq y\}$.

ii. Which of the following a transitive binary relation on $\mathbb{N}$? Explain. (0 or 4 pts.)

i. $\{(x, y) \in \mathbb{N}^2 : 5 \text{ divides } x - y\}$.

ii. $\{(x, y) \in \mathbb{N}^2 : 5 \text{ divides } x + y\}$.

iii. $\{(x, y) \in \mathbb{N}^2 : 5 > x - y\}$.

iv. $\{(x, y) \in \mathbb{N}^2 : 12 < x - y\}$.

iii. Show that the intersection of a set of transitive relations on X is a transitive relation on X. (4 pts.)

iv. Show that for any binary relation R on X the intersection R^t of all the transitive relations that contain R is the unique smallest transitive relation on X that contains R. (10 pts.)

v. Show that if R and S are two binary relations then $(R \cap S)^t \subseteq R^t \cap S^t$. (8 pts.)

vi. Let R be a binary relation. Show that the subset

$$\{S := (x, y) \in X^2 : \exists x = y_1,\, y_2, \ldots, y_n = y \in X \text{ such that}$$
$$(y_i, y_i + 1) \in R \text{ for all } i = 1, \ldots, n-1\}$$

is a transitive relation that contains R. Conclude that $R^t = S$. (10 pts.)

vii. Show that in general $(R \cap S)^t \neq R^t \cap S^t$. (5 pts.)

II. Partial Orders. A binary relation $<$ on a set X is called a **partial order** on X if (writing $x < y$ instead of $(x, y) \in <$),

PO1. Irreflexivity. For every $x \in X$, $x \not< x$.

and

PO2. Transitivity. For every x, y, $z \in X$, if $x < y$ and $y < z$ then $x < z$.

We write $x \leq y$ if either $x < y$ or $x = y$.

Let $(X, <)$ be a partially ordered set and $A \subseteq X$. An element $u \in X$ is called an **upper bound** of A if $a \leq u$ for all $a \in A$. An element $v \in X$ is called a **least upper bound** of A if i) v is an upper bound for A and ii) for any upper bound u of A, if $u \leq v$ then $u = v$.

i. Give an example of a partially ordered set $(X, <)$ and a subset A of X which

   i. has a least upper bound which is not in A.

   ii. has exactly two least upper bounds.

   iii. does not have a least upper bound.

   iv. has a least upper bound which is in A. (4 pts.)

ii. Let $(X, <)$ be a partially ordered set and A a subset of X. Suppose that A has a least upper bound which is in A. Show that this is the only upper bound of A. (2 pts.)

iii. Let $(X, <)$ be a partially ordered set. Show that any element of X is an upper bound of $\emptyset$. (2 pts.)

iv. Let $(X, <)$ be a partially ordered set. What can you say about $(X, <)$ if $\emptyset$ has a least upper bound? (2 pts.)

v. Let U be a set and let $X = \wp(U)$. Order X by inclusion. Show that this is a partial order on X. (2 pts.) Show that any subset of X has a unique least upper bound. (5 pts.)

vi. Let $(X, <)$ be a partial order. Suppose that for any a, $b \in X$, the set $\{a, b\}$ has a unique least upper bound. Let $a \vee b$ denote this least upper bound.

   i. Give an infinite example of such a partially ordered set. (2 pts.)

   ii. Prove or disprove: $(a \vee b) \vee c = a \vee (b \vee c)$ for all a, b, $c \in X$. (10 pts.)

III. Total Orders. If in addition to PO1 and PO2 stated above,

O3 For every x, $y \in X$, either $x < y$ or $x = y$ or $y < x$,

then the partial order is called a **total order**.

i. Show that in a totally ordered set $(X, <)$ if a subset A of X has a least upper bound then this least upper bound is the only upper bound of A. (4 pts.)

IV. Well-Ordered Sets. We say that a totally ordered set $(X, <)$ is a **well-ordered set** (or that $<$ well-orders X) if every nonempty subset of X contains a minimal element for that order, i.e. if for every nonempty subset A of X, there is an $m \in A$ such that $m \leq a$ for all a in A. (Note that the element m must be in A).

i. Give an example of a finite and an infinite well-ordered set. (2 pts.)

ii. Let $X = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\}$. On X define the relation $<$ as follows: For all x, $y \in \mathbb{N}$,

$$\begin{array}{lll} (x,0) < (y,0) & \text{if and only if} & x < y \\ (x,1) < (y,1) & \text{if and only if} & x < y \\ (x,0) < (y,1) & \text{always} & \end{array}$$

   i. Is $(X, <)$ a totally ordered set? (2 pts.)

   ii. Is $(X, <)$ a well-ordered set? (2 pts.)

iii. Is the set $\{1/n : n \in \mathbb{N} \setminus \{0\}\}$ together with the natural order a well-ordered set? (2 pts.)

iv. Is the set $\{1/n : n \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$ together with the natural order a well-ordered set? (2 pts.)

v. Find an infinite well-ordered set with a maximal element. (4 pts.)

vi. Show that in a well-ordered set X, the minimal element of any nonempty subset is unique. (2 pts.)

vii. Show that every nonempty well-ordered set has a unique minimal element. (2 pts.)

viii. Let $(X, <)$ be a well-ordered set. Show that all the elements of X except possibly one of them satisfies the following property: "There exists a y such that $x < y$ and for all z if $x < z$ then $y < z$". (5 pts.) Show that such a y, when exists, is unique (3 pts.)