Set Theory (Math 112)
First Midterm

Spring 2003
Ali Nesin

June 8, 2003

Let X be a set. A **filter** on X is a set Im of subsets of X that satisfies the following properties:

i) If $A \in \mathrm{Im}$ and $A \subseteq B \subseteq X$, then $B \in \mathrm{Im}$.

ii) If A and B are in Im, then so is $A \cap B$.

iii) $\emptyset \notin \mathrm{Im}$ and $X \in \mathrm{Im}$.

If $A \subseteq X$ is a fixed nonempty subset of X, then the set of subsets $\mathrm{Im}(A)$ of X that contain A is a filter on X. Such a filter is called **principal filter**. If X is infinite, then the set of cofinite subsets of X is a filter, called **Fréchet filter**. A filter is called **ultrafilter** if it is a maximal filter.

We fix a set X.

1. Show that a principle filter $\mathrm{Im}(A)$ on X is an ultrafilter if and only if A is a singleton set.

2. Show that the Fréchet filter (on an infinite set) is not contained in a principal filter.

3. Show that the intersection of a set of filters is a filter.

4. Show that if Im is a set of subsets of X such that $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ for any $A_1, A_2, \ldots, A_n \in \mathrm{Im}$, then there is a filter that contains Im. Describe this filter in terms of Im.

5. Show that a filter Im is an ultrafilter if and only if for any $A \subseteq X$, either A or A^c is in Im.

6. Conclude that any ultrafilter on X that contains a finite subset of X is a principal filter. Deduce that every nonprincipal ultrafilter contains the Fréchet filter.

7. (AC) Show that for any filter Im on X, there is an ultrafilter on X that contains Im.

8. (AC) Show that if X is infinite then there are nonprincipal ultrafilters on X.

9. Let Im be a filter on X. Let A_x $(x \in X)$ be sets. Consider the product

$$\begin{array}{rcl} \prod_{x\in X} A_x & := & \{f = (f_x)_{x\in X} : f_x \in A_x \text{ for all } x \in X\} \\ & = & \{f : X \longrightarrow \prod_{x\in X} A_x : f(x) \in A_x \text{ for all } x \in X\}. \end{array}$$

On the set $\prod_{x\in X A_x}$ consider the relation defined by

$$f \equiv g \iff \{x \in X : f(x) = g(x)\} \in \mathrm{Im}\,.$$

Show that this is an equivalence relation.

10. Show that if $A_x = A$ for all $x \in X$ and if Im is nonprincipal filter then the map that sends an element $a \in A$ to the class of the constant function a is an injection from A into M.

11. Suppose that each A_x is a set ordered by a relation $<$. On M define the relation $<$ by,

$$[(f_x)_{x\in X}] < [(g_x)_{x\in X} \iff \{x \in X : f_x < g_x\} \in \mathrm{Im}\,.$$

Show that this defines an order on M.

12. Show that if Im is an ultrafilter and if each $(A_x, <)$ is totally ordered then so is $(M, <)$.

13. Suppose that each $(A_x, <)$ has a largest element. Is it true that $(M, <)$ has a largest element?

14. Suppose that each $(A_x, <)$ is well-ordered. Is it true that $(M, <)$ is well-ordered?