1. Let $G$ be any abelian group. Show that any finite dimensional irreducible representation of $G$ over an algebraically closed field is one dimensional.

Let $K$ be the field and $V$ be the irreducible module. $\dim_K(V) \leq |G| - 1 < \infty$. Let $g \in G$. Since $K$ is algebraically closed, $g$ has an eigenvalue, say $\lambda$. Let $V_\lambda$ be the eigenspace of $g$. Then for all $h \in G$ and $v \in V_\lambda$, $g(hv) = (gh)v = (hg)v = h(gv) = h(\lambda v) = \lambda(hv)$ and so $hv \in V_\lambda$. Therefore $V_\lambda$ is a $G$-space and so $V_\lambda = V$. Being true for all $g \in G$, every element of $G$ acts as a scalar on $V$. Therefore $V$ is one dimensional.

2. Let $M$ be an $R$-module.
   a) Show that $\text{End}_R(M)$ is naturally a ring. [4 pts.]
   b) Show that $M$ is also an $\text{End}_R(M)$-module. [3 pts.]
   c) Assume that $rM = 0$ implies $r = 0$ (i.e. that $\text{Ann}_R(M) = 0$). Show that $R \leq \text{End}_{\text{End}_R(M)}(M)$. [3 pts.]

   **Answer:** a) $\text{End}_R(M)$ is a ring under addition and composition. This is easy to check.
   b) This is also clear.
   c) By definition $\text{End}_R(M)$ are the additive maps that commute with $R$-multiplications, so the $R$-multiplications commute with $\text{End}_R(M)$ as well. The condition “$rM = 0 \Rightarrow r = 0$” just tells that $R$ imbeds in $\text{End}_{\text{End}_R(M)}(M)$.

3. (Schur’s Lemma)
   a) Let $M$ and $M_1$ be two irreducible $R$-modules. Let $\phi : M \longrightarrow M_1$ be a module homomorphism. Show that either $\phi$ is the zero map or an isomorphism. (6 pts.)
   b) Let $M$ be an irreducible $R$-module. Show that $\text{End}_R(M)$ is a division ring. (4 pts.)
Let \( G \) be a finite group and \( V \) an irreducible \( K[G] \)-module. Assume \( K \) is algebraically closed. Show that \( \text{End}_{K[G]}(V) \cong K \). (10 pts.)

Let \( \phi \in \text{End}_{K[G]}(V) \). Since \( G \) is finite, \( V \) is finite dimensional (of dimension at most \(|G| - 1\)). Therefore \( \phi \) has an eigenvalue, say \( \lambda \). Let

\[
V_{\lambda} = \{ v \in V : \phi(v) = \lambda v \}.
\]

For \( g \in G \) and \( v \in V_{\lambda} \), \( \phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv) \) and therefore \( gv \in V_{\lambda} \). It follows that \( V_{\lambda} \) is a \( G \)-space and so \( V = V_{\lambda} \). Hence \( \phi = \lambda \text{Id}_V \).

The same proof works if \( G \) is not necessarily finite but with the additional assumption that \( \dim_K(V) < \infty \). On the other hand if \( \dim_K(V) \) is infinite, this does not hold anymore: Take \( G = \mathbb{Z} = \langle x \rangle \), \( V = \sum_{n \in \mathbb{Z}} Kx^n \) and let \( xv_i = v_{i+1} \). Then \( V \) is an irreducible \( \mathbb{Z} \)-space as it can be checked easily.

5. Find all irreducible representations of \((\mathbb{Z}/2\mathbb{Z})^n\) over any field. (60 pts.)

As we know, any irreducible \( G \)-module is a quotient of \( K[G] \) by a maximal left ideal. Let \( G = (\mathbb{Z}/2\mathbb{Z})^n \). Since

\[
K[G] \cong K[X_1, \ldots, X_n]/\langle X_1^2 - 1, \ldots, X_n^2 - 1 \rangle,
\]

any irreducible \( G \)-module is isomorphic to \( K[X_1, \ldots, X_n]/M \) for some maximal ideal \( M \) of \( K[X_1, \ldots, X_n]/\langle X_1^2 - 1, \ldots, X_n^2 - 1 \rangle \). Since \( (X_i - 1)(X_i + 1) = X_i^2 - 1 \in \langle X_1^2 - 1, \ldots, X_n^2 - 1 \rangle \leq M \) and \( M \) is a maximal ideal (so that \( K[X_1, \ldots, X_n]/M \) is a field, hence \( M \) is a prime ideal), either \( X_i - 1 \) or \( X_i + 1 \) is in \( M \), say \( X_i - \epsilon_i \in M \), \( \epsilon_i = \pm 1 \). Thus \( K[X_1, \ldots, X_n]/M \cong K \) and \( X_i(1) = \epsilon_i \).

Therefore all irreducible representations of \((\mathbb{Z}/2\mathbb{Z})^n\) over a field \( K \) are one-dimensional. They are given by a subset \( A \) of \( \{1, \ldots, n\} \) by the rule \( \phi_A(X_i)(1) = -1 \) if and only if \( i \in A \). It is easy to check that these representations are inequivalent if \( \text{char}(K) \neq 2 \). Thus there are exactly \( 2^n \) inequivalent representations of \((\mathbb{Z}/2\mathbb{Z})^n\) if \( \text{char}(K) \neq 2 \) and there is only one representation (the trivial one) if \( \text{char}(K) = 2 \). They all have dimension 1.