Algebra (Math 212) Final

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1. Let G be any abelian group. Show that any finite dimensional irreducible representation of G over an algebraically closed field is one dimensional. [10 pts.]

   Let K be the field and V be the irreducible module. $\dim_K(V) \leq |G|-1 < \infty$. Let $g \in G$. Since K is algebraically closed, g has an eigenvalue, say λ. Let V_λ be the eigenspace of g. Then for all $h \in G$ and $v \in V_\lambda$, $g(hv) = (gh)v = (hg)v = h(gv) = h(\lambda v) = \lambda(hv)$ and so $hv \in V_\lambda$. Therefore V_λ is a G-space and so $V_\lambda = V$. Being true for all $g \in G$, every element of G acts as a scalar on V. therefore V is one dimensional.

2. Let M be an R-module.

   a) Show that $\mathrm{End}_R(M)$ is naturally a ring. [4 pts.]

   b) Show that M is also an $\mathrm{End}_R(M)$-module. [3 pts.]

   c) Assume that $rM = 0$ implies $r = 0$ (i.e. that $\mathrm{Ann}_R(M) = 0$. Show that $R \leq \mathrm{End}_{\mathrm{End}_R(M)}(M)$. [3 pts.]

   Answer: a) $\mathrm{End}_R(M)$ is a ring under addition and composition. This is easy to check.

   b) This is also clear.

   c) By definition $\mathrm{End}_R(M)$ are the additive maps that commute with R-multiplications, so the R-multiplications commute with $\mathrm{End}_R(M)$ as well. The condition "$rM = 0 \Rightarrow r = 0$" just tells that R imbeds in $\mathrm{End}_{\mathrm{End}_R(M)}(M)$.

3. (Schur's Lemma)

   a) Let M and M_1 be two irreducible R-modules. Let $\phi : M \longrightarrow M_1$ be a module homomorphism. Show that either ϕ is the zero map or an isomorphism. (6 pts.)

   b) Let M be an irreducible R-module. Show that $\mathrm{End}_R(M)$ is a division ring. (4 pts.)

Answer: a) $\phi(M)$ is a a submodule of M_1. Therefore either $\phi(M) = 0$ or $\phi(M) = M_1$. In the first case $\phi = 0$. In the second case ϕ is onto.

$\mathrm{Ker}(\phi)$ is a submodule of M. Therefore either $\mathrm{Ker}(\phi) = 0$ or $\mathrm{Ker}(\phi) = M$. In the first case ϕ is one to one. In the second case $\phi = 0$.

The result follows from these.

b) The fact that $\mathrm{End}_R(M)$ is a ring was part I. It follows from part a that each ϕ is either 0 or an isomorphism. The inverse map of an isomorphism is also an isomorphism.

4. Let G be a finite group and V an irreducible $K[G]$-module. Assume K is algebraically closed. Show that $\mathrm{End}_{K[G]}(V) \simeq K$. (10 pts.)

   Let $\phi \in \mathrm{End}_{K[G]}(V)$. Since G is finite, V is finite dimensional (of dimension at most $|G| - 1$). Therefore ϕ has an eigenvalue, say λ. Let

   $$V_\lambda = \{v \in V : \phi(v) = \lambda v\}.$$

   For $g \in G$ and $v \in V_\lambda$, $\phi(gv) = g\phi(v) = g(\lambda v) = \lambda(gv)$ and therefore $gv \in V_\lambda$. It follows that V_λ is a G-space and so $V = V_\lambda$. Hence $\phi = \lambda \, \mathrm{Id}_V$.

   The same proof works if G is not necessarily finite but with the additional assumption that $\dim_K(V) < \infty$. On the other hand if $\dim_K(V)$ is infinite, this does not hold anymore: Take $G = \mathbb{Z} = \langle x \rangle$, $V = \sum_{i \in \mathbb{Z}} K v_i$ and let $x v_i = v_{i+1}$. Then V is an irreducible $\mathbb{Z}$-space as it can be checked easily.

5. Find all irreducible representations of $(\mathbb{Z}/2\mathbb{Z})^n$ over any field. (60 pts.)

   As we know, any irreducible G-module is a quotient of $K[G]$ by a maximal left ideal. Let $G = (\mathbb{Z}/2\mathbb{Z})^n$. Since

   $$K[G] \simeq K[X_1, \ldots, X_n]/\langle X_1^2 - 1, \ldots, X_n^2 - 1\rangle,$$

   any irreducible G-module is isomorphic to $K[X_1, \ldots, X_n]/M$ for some maximal ideal M of $K[X_1, \ldots, X_n]/M$ containing $\langle X_1^2 - 1, \ldots, X_n^2 - 1\rangle$. Since $(X_i - 1)(X_i + 1) = X_i^2 - 1 \in \langle X_1^2 - 1, \ldots, X_n^2 - 1\rangle \leq M$ and M is a maximal ideal (so that $K[X_1, \ldots, X_n]/M$ is a field, hence M is a prime ideal), either $X_i - 1$ or $X_i + 1$ is in M, say $X_i - \epsilon_i \in M$, ($\epsilon_i = \pm 1$). Thus $K[X_1, \ldots, X_n]/M \simeq K$ and $X_i(1) = \epsilon_i$.

   Therefore all irreducible representations of $(\mathbb{Z}/2\mathbb{Z})^n$ over a field K are one-dimensional. They are given by a subset A of $\{1, \ldots, n\}$ by the rule $\phi_A(X_i)(1) = -1$ if and only if $i \in A$. It is easy to check that these representations are inequivalent if $\mathrm{char}(K) \neq 2$. Thus there are exactly 2^n inequivalent representations of $(\mathbb{Z}/2\mathbb{Z})^n$ if $\mathrm{char}(K) \neq 2$ and there is only one representation (the trivial one) if $\mathrm{char}(K) = 2$. They all have dimension 1.