Set Theory

Midterm
May 2003
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Let X be a set. A **filter** on X is a set $\Im$ of subsets of X that satisfies the following properties:

i) If $A \in \Im$ and $A \subseteq B \subseteq X$, then $B \in \Im$.

ii) If A and B are in $\Im$, then so is $A \cap B$.

iii) $\emptyset \notin \Im$ and $X \in \Im$.

If $A \subseteq X$ is a fixed nonempty subset of X, then the set of subsets $\Im(A)$ of X that contain A is a filter on X. Such a filter is called **principal filter**.

If X is infinite, then the set of cofinite subsets of X is a filter, called **Fréchet filter**.

A filter is called **ultrafilter** if it is a maximal filter.

We fix a set X.

1. Show that a principle filter $\Im(A)$ on X is an ultrafilter iff A is a singleton set.
2. Show that the Fréchet filter (on an infinite set) is not principal.
3. Show that the intersection of a set of filters is a filter.
4. Show that if $\Im$ is a set of subsets of X such that $A_1 \cap A_2 \cap ... \cap A_n \neq \emptyset$ for any $A_1, A_2, ..., A_n \in \Im$, then there is a filter that contains $\Im$. Describe this filter in terms of $\Im$.
5. Show that a filter $\Im$ is an ultrafilter if and only if for any $A \subseteq X$, either A or A^c is in $\Im$.
6. Conclude that any ultrafilter on X that contains a finite subset of X is a principal filter. Deduce that every nonprinciple ultrafilter contains the Fréchet filter.
7. (AC) Show that for any filter $\Im$ on X, there is an ultrafilter on X that contains $\Im$.
8. (AC) Show that if X is infinite then there are nonprincipal ultrafilters on X.
9. Let $\Im$ be a filter on X. Let A_x ($x \in X$) be sets. Consider the product

$$\Pi_{x \in X} A_x := \{f = (f_x)_{x \in X} : f_x \in A_x \text{ for all } x \in X\} = \{f : X \to \cup_{x \in X} A_x : f(x) \in A_x\}.$$

On $\Pi_{x \in X} A_x$ consider the relation $\equiv$ defined by

$$f \equiv g \Leftrightarrow \{x \in X : f(x) = g(x)\} \in \Im.$$

Show that this is an equivalence relation. From now on we fix a filter $\Im$ on X and we let $M = \Pi_{x \in X} A_x/\equiv$. If $f \in \Pi_{x \in X} A_x$, we let $[f]$ denote the class of f in M.

10. Show that if $A_x = A$ for all $x \in X$ and if $\Im$ is nonprincipal filter then the map that sends an element $a \in A$ to the class of the constant function a is an injection from A into M.
11. Suppose that each A_x is a set ordered by a relation $<$. On M define the relation $<$ by,

$$[(f_x)_{x \in X}] < [(g_x)_{x \in X}] \Leftrightarrow \{x \in X : f_x < g_x\} \in \Im.$$

Show that this defines an order on M.

12. Show that if $\Im$ is an ultrafilter and each $(A_x, <)$ is totally ordered then so is $(M, <)$.
13. Suppose that each $(A_x, <)$ has a largest element. Is it true that $(M, <)$ has a largest element?
14. Suppose that each $(A_x, <)$ is well-ordered. Is it true that $(M, <)$ is well-ordered?