Math 111. Set Theory
Resit – Solutions

Ali Nesin
March 17, 2003

Let $X$ be a set. A set of subsets of $X$ - whose elements are called open subsets of $X$ - is called a topology on $X$, if

T1. $\emptyset$ and $X$ are open subsets of $X$.
T2. The intersection of two open subsets of $X$ is an open subset of $X$.
T3. The union of any collection of open subsets of $X$ is an open subset of $X$.

More formally, a subset $\tau$ of $\mathcal{P}(X)$ is called a topology on $X$, if

T1. $\emptyset \in \tau$ and $X \in \tau$.
T2. If $U, V \in \tau$, then $U \cap V \in \tau$.
T3. If $\sigma \subseteq \tau$, then $\bigcup \sigma \in \tau$.

i. Show that $\{\emptyset, X\}$ is a topology on $X$. (2 pts.)

Answer. We must check that the set $\{\emptyset, X\}$ satisfies T1, T2 and T3. T1 is clear. T2 and T3, in this context, just mean that the set $\{\emptyset, X\}$ is closed under intersection and union. This is clear as well.

ii. Show that $\wp(X)$ is a topology on $X$. (2 pts.)

Answer. Clearly $\emptyset, X \in \wp(X)$. T2 and T3 are clear as well.

iii. Let $A$ be a subset of $X$. Show that $\{\emptyset, A, X\}$ is a topology on $X$. (2 pts.)

Answer. Clear.

iv. Let $A$ and $B$ be two subsets of $X$. Find a finite topology on $X$ that contains $A$ and $B$. (3 pts.)

v. Let $A$, $B$ and $C$ be three subsets of $X$. Find a finite topology on $X$ that contains $A$ and $B$. What is the maximum size of the smallest such topology? (4 pts.)

vi. Let $\tau$ be a topology on $X$. Let $A \subseteq X$. Show that there is a unique largest open subset of $A$. We denote this subset by $A^\circ$. (6 pts.)

Show that for all subsets $A$ and $B$ of $X$,

a) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$. (4 pts.)
b) \((A \cap B)^\circ \subseteq A^\circ \cap B^\circ\). (5 pts.)
c) \((A \cup B)^\circ = A^\circ \cup B^\circ\). (7 pts.)

vii. Show that if \(\sigma\) and \(\tau\) are topologies on \(X\), then \(\sigma \cap \tau\) is also a topology on \(X\). (5 pts.)

viii. Show that if \(T\) is any set of topologies on \(X\), then \(\cap T\) is also a topology on \(X\). (3 pts.)

ix. Show that if \(\sigma \subseteq \wp(X)\) is a set of subsets of \(X\), then the intersection \(\langle \sigma \rangle\) of all the topologies that contain \(\sigma\) is the smallest topology on \(X\) that contains \(\sigma\). In other words, if \(A \in \sigma\) then \(A\) is open in the topology \(\langle \sigma \rangle\), and \(\langle \sigma \rangle\) is the smallest topology on \(X\) with this property. This topology is called the **topology generated** by \(\sigma\). (10 pts.)

x. Let \(\sigma\) be the set of singleton subsets of \(X\). Find \(\langle \sigma \rangle\). (3 pts.)

xi. Let \(n \in \mathbb{N}\). Let \(\sigma\) be the set of subsets of \(X\) of cardinality \(n\). Find \(\langle \sigma \rangle\). (5 pts.)

xii. Assume \(\sigma\) is the set of cofinite subsets of \(X\). Find \(\langle \sigma \rangle\). (4 pts.)

xiii. Let \(\sigma \subseteq \wp(X)\). Consider the set \([\sigma]\) of subsets \(A\) of \(X\) with the following property:

   For any \(x \in A\) there are finitely many \(S_1, \ldots, S_n \in \sigma\) such that
   \[
   x \in S_1 \cap \cdots \cap S_n \subseteq U.
   \]

   a) Show that \([\sigma]\) is a topology on \(X\). (5 pts.)
   b) Show that \(\langle \sigma \rangle \subseteq [\sigma]\). (5 pts.)
   c) Show that \(\langle \sigma \rangle = [\sigma]\). (5 pts.)

   Thus \(U \in \langle \sigma \rangle\) if and only if for all \(x \in \langle \sigma \rangle\) there are finitely many \(S_1, \ldots, S_n \in \sigma\) such that \(x \in S_1 \cap \cdots \cap S_n \subseteq U\). (15 pts.)

xiv. Let \(\sigma\) be the set of all open intervals of \(\mathbb{R}\). Consider the set \(\mathbb{R}\) with the topology \(\langle \sigma \rangle\).

   a) Show that a subset \(A\) of \(\mathbb{R}\) is open (in this topology) if and only if for all \(a \in A\) there is an \(\epsilon \in \mathbb{R}^>0\) such that \((a - \epsilon, a + \epsilon) \subseteq A\). (7 pts.)
   b) Show that no finite and nonempty subset of \(\mathbb{R}\) is open (in this topology). (2 pts.)
   c) Show that \([0, 1)\) is not open. (3 pts.)
   d) Show that a cofinite subset is open. (3 pts.)
   d) Is \(\mathbb{Q}\) open? (3 pts.)
xv. Let $\sigma_1$ be the set of all intervals of $\mathbb{R}$ of the form $[a, b)$ of $\mathbb{R}$ for $a \leq b \in \mathbb{R}$.

a. Show that $\langle \sigma \rangle \subset \langle \sigma_1 \rangle$. (5 pts.)

b. Can a singleton set be open in this topology? (3 pts.)

c. Show that $[0, 1]$ is not open in this topology. (3 pts.)

xvi. Let $\sigma_2$ be the set of all closed and bounded intervals of $\mathbb{R}$. Show that $\langle \sigma_1 \rangle \subset \langle \sigma_2 \rangle$ where $\sigma_1$ is as in number xvi. (5 pts.)