Math 111. Set Theory
Resit – Solutions

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Let X be a set. A set of subsets of X - whose elements are called **open subsets** of X - is called a **topology** on X, if

T1. $\emptyset$ and X are open subsets of X.

T2. The intersection of two open subsets of X is an open subset of X.

T3. The union of any collection of open subsets of X is an open subset of X.

More formally, a subset τ of $\wp(X)$ is called a topology on X, if

T1. $\emptyset \in \tau$ and $X \in \tau$.
T2. If $U, V \in \tau$, then $U \cap V \in \tau$.
T2. If $\sigma \subseteq \tau$, then $\cup\sigma \in \tau$.

i. Show that $\{\emptyset, X\}$ is a topology on X. (2 pts.)

Answer. We must check that the set $\{\emptyset, X\}$ satisfies T1, T2 and T3. T1 is clear. T2 and T3, in this context, just mean that the set $\{\emptyset, X\}$ is closed under intersection and union. This is clear as well.

ii. Show that $\wp(X)$ is a topology on X. (2 pts.)

Answer. Clearly $\emptyset$, $X \in \wp(X)$. T2 and T3 are clear as well.

iii. Let A be a subset of X. Show that $\{\emptyset, A, X\}$ is a topology on X. (2 pts.)

Answer. Clear.

iv. Let A and B be two subsets of X. Find a finite topology on X that contains A and B. (3 pts.)

v. Let A, B and C be three subsets of X. Find a finite topology on X that contains A and B. What is the maximum size of the smallest such topology? (4 pts.)

vi. Let τ be a topology on X. Let $A \subseteq X$. Show that there is a unique largest open subset of A. We denote this subset by A°. (6 pts.)

Show that for all subsets A and B of X,

a) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$. (4 pts.)

b) $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. (5 pts.)

c) $(A \cup B)^\circ = A^\circ \cup B^\circ$. (7 pts.)

vii. Show that if σ and τ are topologies on X, then $\sigma \cap \tau$ is also a topology on X. (5 pts.)

viii. Show that if T is any set of topologies on X, then $\cap T$ is also a topology on X. (3 pts.)

ix. Show that if $\sigma \subseteq \wp(X)$ is a set of subsets of X, then the intersection $\langle\sigma\rangle$ of all the topologies that contain σ is the smallest topology on X that contains σ. In other words, if $A \in \sigma$ then A is open in the topology $\langle\sigma\rangle$, and $\langle\sigma\rangle$ is the smallest topology on X with this property. This topology is called the **topology generated** by σ. (10 pts.)

x. Let σ be the set of singleton subsets of X. Find $\langle\sigma\rangle$. (3 pts.)

xi. Let $n \in \mathbb{N}$. Let σ be the set of subsets of X of cardinality n. Find $\langle\sigma\rangle$. (5 pts.)

xii. Assume σ is the set of cofinite subsets of X. Find $\langle\sigma\rangle$. (4 pts.)

xiii. Let $\sigma \subseteq \wp(X)$. Consider the set $[\sigma]$ of subsets A of X with the following property:

For any $x \in A$ there are finitely many $S_1, \ldots, S_n \in \sigma$ such that $x \in S_1 \cap \ldots \cap S_n \subseteq U$.

a) Show that $[\sigma]$ is a topology on X. (5 pts.)

b) Show that $\langle\sigma\rangle \subseteq [\sigma]$. (5 pts.)

c) Show that $\langle\sigma\rangle = [\sigma]$. (5 pts.)

Thus $U \in \langle\sigma\rangle$ if and only if for all $x \in \langle\sigma\rangle$ there are finitely many $S_1, \ldots, S_n \in \sigma$ such that $x \in S_1 \cap \ldots \cap S_n \subseteq U$. (15 pts.)

xiv. Let σ be the set of all open intervals of $\mathbb{R}$. Consider the set $\mathbb{R}$ with the topology $\langle\sigma\rangle$.

a) Show that a subset A of $\mathbb{R}$ is open (in this topology) if and only if for all $a \in A$ there is an $\epsilon \in \mathbb{R}^{>0}$ such that $(a - \epsilon, a + \epsilon) \subseteq A$. (7 pts.)

b) Show that no finite and nonempty subset of $\mathbb{R}$ is open (in this topology). (2 pts.)

c) Show that $[0, 1)$ is not open. (3 pts.)

d) Show that a cofinite subset is open. (3 pts.)

d) Is $\mathbb{Q}$ open? (3 pts.)

xv. Let σ_1 be the set of all intervals of $\mathbb{R}$ of the form $[a, b)$ of $\mathbb{R}$ for $a \leq b \in \mathbb{R}$.

   a. Show that $\langle \sigma \rangle \subset \langle \sigma_1 \rangle$. (5 pts.)

   b. Can a singleton set be open in this topology? (3 pts.)

   c. Show that $[0, 1]$ is not open in this topology. (3 pts.)

xvi. Let σ_2 be the set of all closed and bounded intervals of $\mathbb{R}$. Show that $\langle \sigma_1 \rangle \subset \langle \sigma_2 \rangle$ where σ_1 is as in number **??**. (5 pts.)