# Naive Set Theory (Math 111)
# First Midterm
# Questions and Answers

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1. a) *Given a set $X$, define $\cup X$ as follows:*

$$y \in \cup X \textit{ if and only if there is an } x \in X \textit{ such that } y \in x.$$

*Show that $\cup\emptyset = \emptyset$.* (3 pts.)

**Proof:** Set $X = \emptyset$ in the definition. Thus

$$y \in \cup\emptyset \text{ if and only if there is an } x \in \emptyset \text{ such that } y \in x.$$

Since there is no $x \in \emptyset$, we see that $\cup\emptyset = \emptyset$.

b) *Given a set $X$, define $\cap_1 X$ and $\cap_2 X$ as follows:*

$$\begin{array}{lll} y \in \cap_1 X & \textit{if and only if} & y \in x \textit{ for all } x \in X \\ y \in \cap_2 X & \textit{if and only if} & y \in \cup X \textit{ and } y \in x \textit{ for all } x \in X \end{array}$$

*Is $\cap_1 X = \cap_2 X$ for all $X$? Which definition do you prefer for $\cap X$ and why?* (5 pts.)

**Answer:** Certainly $\cap_2 X \subseteq \cap_1 X$ because $\cap_2 X = (\cap_1 X) \cap (\cup X)$ by definition.

The reverse inclusion holds only if $X \neq \emptyset$. Indeed, assume $X \neq \emptyset$ and let $y \in \cap_1 X$. To show that $y \in \cap_2 X$, we need to show that $y \in \cup X$, i.e. that $y \in x$ for some $x \in X$. Since $X \neq \emptyset$ and $y \in \cap_1 X$, this holds trivially, i.e. $y \in x$ for some $x \in X$.

The reverse inclusion does not hold if $X = \emptyset$. Indeed, $\cap_2\emptyset = (\cap_1\emptyset) \cap (\cup\emptyset) = (\cap_1\emptyset) \cap \emptyset = \emptyset$. On the other hand,

$$y \in \cap_1\emptyset \text{ if and only if } y \in x \text{ for all } x \in \emptyset$$

and this condition holds for all $y$. Hence $\cap_1\emptyset$ is the whole universe and is not even a set.

Thus we should prefer $\cap_2 X$ for the definition of $\cap X$ since the outcome would then be a set for any set $X$, even for $X = \emptyset$!

2. *Find a set $X$ such that $X \cap \wp(X) \neq \emptyset$.* (3 pts.)

   **Answer.** Since $\emptyset \in \wp(X)$ for any $X$, any set $X$ that contains $\emptyset$ as element would do, e.g. we may take $X = \{\emptyset\}$.

3. *Let $(A_i)_{i \in I}$ be a family of sets.*

   a) *Show that $\bigcap_{i \in I} \wp(A_i) = \wp(\bigcap_{i \in I} A_i)$.* (3 pts.)

   **Proof:**

$$\begin{array}{lll} X \in \bigcap_{i \in I} \wp(A_i) & \text{if and only if} & X \in \wp(A_i) \text{ for all } i \in I \\ & \text{if and only if} & X \subseteq A_i \text{ for all } i \in I \\ & \text{if and only if} & X \subseteq \bigcap_{i \in I} A_i \\ & \text{if and only if} & X \in \wp(\bigcap_{i \in I} A_i) \end{array}$$

   b) *What is the relationship between $\bigcup_{i \in I} \wp(A_i)$ and $\wp(\bigcup_{i \in I} A_i)$?* (4 pts.)

   **Answer:** We have $\bigcup_{i \in I} \wp(A_i) \subseteq \wp(\bigcup_{i \in I} A_i)$. Indeed,

$$\begin{array}{lll} X \in \bigcup_{i \in I} \wp(A_i) & \text{if and only if} & X \in \wp(A_i) \text{ for some } i \in I \\ & \text{if and only if} & X \subseteq A_i \text{ for some } i \in I \\ & \text{implies} & X \subseteq \cup_{i \in I} A_i \\ & \text{if and only if} & X \in \wp(\cup_{i \in I} A_i) \end{array}$$

   The reverse inclusion is false, in fact if one of the sets $A_i$ does not contain all the others, then $\cup_{i \in I} A_i \in \wp(\cup_{i \in I} A_i) \setminus \bigcup_{i \in I} \wp(A_i)$.

4. *What is the set $(X \times Y) \cap (Y \times X)$?* (3 pts.)

   **Answer:** Let $(u, v) \in (X \times Y) \cap (Y \times X)$. Since $(u, v) \in X \times Y$, $u$ is an element of $X$ and $v$ is an element of $Y$. Similarly, since $(u, v) \in Y \times X$, $u$ is an element of $Y$ and $v$ is an element of $X$. Thus both $u$ and $v$ are elements of $X \cap Y$, i.e. $(u, v) \in (X \cap Y) \times (X \cap Y)$.

   The reverse inclusion $(X \cap Y) \times (X \cap Y) \subseteq (X \times Y) \cap (Y \times X)$ is trivial.

5. *Let $\alpha$ be a set such that $x \subseteq \alpha$ for all $x \in \alpha$. Show that $\alpha \cup \{\alpha\}$ has the same property. Give four examples of such sets.* (4 pts.)

   **Proof:** Let $x \in \alpha \cup \{\alpha\}$. Then either $x \in \alpha$ or $x \in \{\alpha\}$.

   If $x \in \alpha$, then by assumption $x \subseteq \alpha$. Since $\alpha \subseteq \alpha \cup \{\alpha\}$, in this case we get $x \subseteq \alpha \cup \{\alpha\}$.

   If $x \in \{\alpha\}$, then $x = \alpha$, and so $x = \alpha \subseteq \alpha \cup \{\alpha\}$.

   Since $\emptyset$ has the property stated, starting from $\emptyset$, we can get as many examples as we wish to, here are the first four:

$$\begin{array}{l} \emptyset = 0 \\ 0 \cup \{0\} = 1 \\ 1 \cup \{1\} = 2 \\ 2 \cup \{2\} = 3 \end{array}$$

6. *Let $\Gamma$ be a graph such that for any vertices $\alpha$, $\alpha_1$, $\beta$, $\beta_1$, if $\alpha \neq \alpha_1$ and $\beta \neq \beta_1$, then there is a $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(\alpha) = \beta$ and $\phi(\alpha_1) = \beta_1$. What can you say about $\Gamma$?* (5 pts.)

   **Answer:** Then either the graph is the complete graph (all possible edges exist) or the graph without edges at all. Indeed, otherwise we may find $\alpha$, $\alpha_1$ and $\beta \neq \beta_1$ such that $\alpha$ and $\alpha_1$ are connected (hence $\alpha \neq \alpha_1$) and $\beta$ and $\beta_1$ are not connected, but then it is impossible to send the connected pair $(\alpha, \alpha_1)$ to the nonconnected pair $(\beta, \beta_1)$.

7. *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a* **one to one** *map such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(x^2) = \phi(x)^2$ for all $x$, $y \in \mathbb{R}$. Show that $\phi(xy) = \phi(x)\phi(y)$ for all $x$, $y \in \mathbb{R}$ and $\phi(q) = q$ for all $q \in \mathbb{Q}$.* (10 pts.)

   **Proof:** For any $x, y \in \mathbb{R}$, we have $\phi(x)^2 + 2\phi(x)\phi(y) + \phi(y)^2 = (\phi(x) + \phi(y))^2 = \phi(x+y)^2 = \phi((x+y)^2) = \phi(x^2 + 2xy + y^2) = \phi(x^2) + 2\phi(xy) + \phi(y^2) = \phi(x)^2 + 2\phi(xy) + \phi(y)^2$ and so $2\phi(x)\phi(y) = 2\phi(xy)$ and simplifying, we get $\phi(x)\phi(y) = \phi(xy)$. This proves the first part.

   Since $\phi(0) = \phi(0+0) = \phi(0) + \phi(0)$, we must have $\phi(0) = 0$.

   Since $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$, we must have $\phi(1) = 0$ or $\phi(1) = 1$. But the first case is forbidden because $\phi$ is one to one and $\phi(0) = 0$ already. Hence $\phi(1) = 1$.

   Now, it follows easily by induction that $\phi(n) = n$ for all $n \in \mathbb{N}$ because $\phi(n+1) = \phi(n) + \phi(1) = \phi(n) + 1 = n+1$ (the last equality is the inductive hypothesis).

   Also, for $n \in \mathbb{N}$, we have $0 = \phi(0) = \phi(n + (-n)) = \phi(n) + \phi(-n)$ and so $\phi(-n) = -\phi(n) = -n$. Thus $\phi(n) = n$ for all $n \in \mathbb{Z}$.

   Now if $q \in \mathbb{Q}$, then $q = n/m$ some $n$, $m \in \mathbb{Z}$ and $m \neq 0$. Then we have $n = \phi(n) = \phi(mn/m) = \phi(m)\phi(n/m) = m\phi(n/m)$ and so $\phi(n/m) = n/m$, i.e. $\phi(q) = q$.

8. *Given a set $X$, define $\wp^n(X)$ as follows by induction on $n$: $\wp^0(X) = X$ and $\wp^{n+1}(X) = \wp(\wp^n(X))$.*

   a) *Is there a natural number $n$ such that for any set $X$, $\{\{\emptyset\}, \{\{X\}\}\} \in \wp^n(X)$?* (8 pts.)

   **Answer:** For $n \geq 4$ note the equivalence of the following propositions:

   $\{\{\emptyset\}, \{\{X\}\}\} \in \wp^n(X)$
   $\{\{\emptyset\}, \{\{X\}\}\} \subseteq \wp^{n-1}(X)$
   $\{\emptyset\}$, $\{\{X\}\} \in \wp^{n-1}(X)$
   $\{\emptyset\}$, $\{\{X\}\} \subseteq \wp^{n-2}(X)$
   $\emptyset$, $\{X\} \in \wp^{n-2}(X)$
   $\emptyset \in \wp^{n-2}(X)$ and $\{X\} \subseteq \wp^{n-3}(X)$
   $\{X\} \subseteq \wp^{n-3}(X)$
   $\{X\} \subseteq \wp^{n-3}(X)$
   $X \in \wp^{n-3}(X)$
   $X \subseteq \wp^{n-4}(X)$

If $n = 4$, the last condition holds for all $X$.

Does it hold for $n = 5$, i.e. do we have $X \subseteq \wp(X)$ for all $X$? For this condition to hold, we need any element of $X$ to be a subset of $X$, and this does not always hold. For any $i \geq 1$, one can find a set $X$ such that $X \not\subseteq \wp^i(X)$. Thus the condition does not hold for any $n \geq 5$ (details are left as an exercise).

For $n = 0, 1, 2, 3$, find examples of $X$ such that $\{\{\emptyset\}, \{\{X\}\}\} \notin \wp^n(X)$.

b) *Show that $\wp(\wp^n(X)) = \wp^n(\wp(X))$ for all sets $X$ and all natural numbers $n$.* (8 pts.)

**Proof:** We proceed by induction on $n$. The condition certainly holds for $n = 0$. Assume it holds for $n$. We have $\wp(\wp^{n+1}(X)) = \wp(\wp^n(\wp(X))) = \wp^{n+1}(\wp(X))$.

c) Show that $\wp^n(\wp^m(X)) = \wp^m(\wp^n(X))$ for all sets $X$ and all natural numbers $n$ and $m$. (8 pts.)

**Proof:** We proceed by induction on $m$. The condition certainly holds for $m = 0$. By part (b) it also holds for $m = 1$. Assume it holds for $m$. We have $\wp^n(\wp^{m+1}(X)) = \wp^n(\wp(\wp^m(X))) = \wp(\wp^n(\wp(X)) = \wp^{n+1}(\wp^m(X))$.

9. *Define a partial order $\prec$ on $\mathbb{N} \setminus \{0, 1\}$ by $x \prec y$ if and only if $x^2 | y$. Describe all the automorphisms of this poset.* (5 pts.)

   **Answer:** The minimal elements of this ordered set (call it $\Gamma$) are the square free numbers. Thus any automorphism of $\Gamma$ should send the square free numbers onto the square free numbers. But the prime numbers have a privilege. Indeed if $p$ is a prime number, then $p$ has an immediate successor (namely $p^2$) that has only one predecessor, namely $p$. Thus any automorphism should be multiplicative and be given by a permutation of primes.

10. *Let $X$ be a set. Let $\Gamma$ be the set of subsets of $X$ with two elements. On $\Gamma$ define the relation $\alpha R \beta$ if and only if $\alpha \cap \beta = \emptyset$. Then $\Gamma$ becomes a graph with this relation.*

    a) *Calculate* $\mathrm{Aut}(\Gamma)$ *when* $|X| = 4$. (3 pts.)

    **Answer:** The graph $\Gamma$ is just six vertices joined two by two. A group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ preserves the edges. And $\mathrm{Sym}(3)$ permutes the edges. Thus the group has $8 \times 3! = 48$ elements.

    More formally, one can prove this as follows. Let the points be $\{1, 2, 3, 4, 5, 6\}$ and the edges be $v_1 = (1, 4)$, $v_2 = (2, 5)$ and $v_3 = (3, 6)$. We can embed $\mathrm{Sym}(3)$ in $\mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(6)$ via

$$\begin{array}{lcl}
\mathrm{Id}_3 & \mapsto & \mathrm{Id}_6 \\
(12) & \mapsto & (12)(45) \\
(13) & \mapsto & (13)(46) \\
(23) & \mapsto & (23)(56) \\
(123) & \mapsto & (123)(456) \\
(132) & \mapsto & (132)(465)
\end{array}$$

For any $\phi \in \mathrm{Aut}(\Gamma)$ there is an element $\alpha$ in the image of $\mathrm{Sym}(3)$ such that $\alpha^{-1}\phi$ preserves the three edges $v_1 = (1,4)$, $v_2 = (2,5)$ and $v_3 = (3,6)$. Thus $\alpha^{-1}\phi \in \mathrm{Sym}\{1,4\} \times \mathrm{Sym}\{2,5\} \times \mathrm{Sym}\{3,6\} \simeq (\mathbb{Z}/2\mathbb{Z})^3$. It follows that $\mathrm{Aut}(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{Sym}(3)$ (to be explained next year).

b) *Draw the graph* $\Gamma$ *when* $X = \{1,2,3,4,5\}$. (3 pts.)

There are ten points. Draw two pentagons one inside the other. Label the outside points as $\{1,2\}$, $\{3,4\}$, $\{5,1\}$, $\{2,3\}$, $\{4,5\}$. Complete the graph.

c) *Show that* $\mathrm{Sym}(5)$ *imbeds in* $\mathrm{Aut}(\Gamma)$ *naturally. (You have to show that each element* $\sigma$ *of* $\mathrm{Sym}(5)$ *gives rise to an automorphism* $\tilde{\sigma}$ *of* $\Gamma$ *in such a way that the map* $\sigma \mapsto \tilde{\sigma}$ *is an injection from* $\mathrm{Sym}(5)$ *into* $\mathrm{Aut}(\Gamma)$ *and that* $\widetilde{\sigma_1 \circ \sigma_2} = \widetilde{\sigma_1} \circ \widetilde{\sigma_2}$*).* (8 pts.)

d) *Show that* $\mathrm{Aut}(\Gamma) \simeq \mathrm{Sym}(5)$. (12 pts.)

**Proof of (c) and (d):** Clearly any element of $\sigma \in \mathrm{Sym}(5)$ gives rise to an automorphism $\tilde{\sigma}$ of $\Gamma$ via $\tilde{\sigma}\{a,b\} = \{\sigma(a), \sigma(b)\}$. The fact that this map preserves the incidence relation is clear. This map is one to one because if $\tilde{\sigma} = \tilde{\tau}$, then for all distinct $a, b, c$, we have $\{\sigma(b)\} = \{\sigma(a), \sigma(b)\} \cap \{\sigma(b), \sigma(c)\} = \tilde{\sigma}\{a,b\} \cap \tilde{\sigma}\{b,c\} = \tilde{\tau}\{a,b\} \cap \tilde{\tau}\{b,c\} = \{\tau(a), \tau(b)\} \cap \{\tau(b), \tau(c)\} = \{\tau(b)\}$ and hence $\sigma(b) = \tau(b)$.

Let $\phi \in \mathrm{Aut}(\Gamma)$. We will compose $\phi$ by elements of $\mathrm{Sym}(5)$ to obtain the identity map. There is an $\sigma \in \mathrm{Sym}(5)$ such that $\phi\{1,2\} = \tilde{\sigma}\{1,2\}$ and $\phi\{3,4\} = \tilde{\sigma}\{3,4\}$. Thus, replacing $\phi$ by $\sigma^{-1}\phi$, we may assume that $\phi$ fixes the vertices $\{1,2\}$ and $\{3,4\}$. Now $\phi$ must preserve or exchange the vertices $\{3,5\}$ and $\{4,5\}$. By applying the element $(34)$ of $\mathrm{Sym}(5)$ we may assume that these two vertices are fixed as well. Now $\phi$ must preserve or exchange the vertices $\{1,3\}$ and $\{2,3\}$. By applying the element $(12)$ of $\mathrm{Sym}(5)$ we may assume that these two vertices are fixed as well. Now all the vertices must be fixed.