1. a) Given a set $X$, define $\cup X$ as follows:

$$y \in \cup X \text{ if and only if there is an } x \in X \text{ such that } y \in x.$$ 

Show that $\cup \emptyset = \emptyset$. (3 pts.)

**Proof:** Set $X = \emptyset$ in the definition. Thus

$$y \in \cup \emptyset \text{ if and only if there is an } x \in \emptyset \text{ such that } y \in x.$$ 

Since there is no $x \in \emptyset$, we see that $\cup \emptyset = \emptyset$.

b) Given a set $X$, define $\cap_1 X$ and $\cap_2 X$ as follows:

$$y \in \cap_1 X \text{ if and only if } y \in x \text{ for all } x \in X$$

$$y \in \cap_2 X \text{ if and only if } y \in \cup X \text{ and } y \in x \text{ for all } x \in X$$

Is $\cap_1 X = \cap_2 X$ for all $X$? Which definition do you prefer for $\cap X$ and why? (5 pts.)

**Answer:** Certainly $\cap_2 X \subseteq \cap_1 X$ because $\cap_2 X = (\cap_1 X) \cap (\cup X)$ by definition.

The reverse inclusion holds only if $X \neq \emptyset$. Indeed, assume $X \neq \emptyset$ and let $y \in \cap_1 X$. To show that $y \in \cap_2 X$, we need to show that $y \in \cup X$, i.e. that $y \in x$ for some $x \in X$. Since $X \neq \emptyset$ and $y \in \cap_1 X$, this holds trivially, i.e. $y \in x$ for some $x \in X$.

The reverse inclusion does not hold if $X = \emptyset$. Indeed, $\cap_2 \emptyset = (\cap_1 \emptyset) \cap (\cup \emptyset) = (\cap_1 \emptyset) \cap \emptyset = \emptyset$. On the other hand,

$$y \in \cap_1 \emptyset \text{ if and only if } y \in x \text{ for all } x \in \emptyset$$

and this condition holds for all $y$. Hence $\cap_1 \emptyset$ is the whole universe and is not even a set.
Thus we should prefer \( \cap_{2}X \) for the definition of \( \cap X \) since the outcome would then be a set for any set \( X \), even for \( X = \emptyset \).

2. Find a set \( X \) such that \( X \cap \wp(X) \neq \emptyset \). (3 pts.)

**Answer:** Since \( \emptyset \in \wp(X) \) for any \( X \), any set \( X \) that contains \( \emptyset \) as element would do, e.g. we may take \( X = \{ \emptyset \} \).

3. Let \( (A_{i})_{i \in I} \) be a family of sets.
   a) Show that \( \bigcap_{i \in I} \wp(A_{i}) = \wp(\bigcap_{i \in I} A_{i}) \). (3 pts.)

**Proof:**

\[
X \in \bigcap_{i \in I} \wp(A_{i}) \quad \text{if and only if} \quad X \in \wp(A_{i}) \text{ for all } i \in I \\
\text{if and only if} \quad X \subseteq A_{i} \text{ for all } i \in I \\
\text{if and only if} \quad X \subseteq \bigcap_{i \in I} A_{i} \\
\text{if and only if} \quad X \in \wp(\bigcap_{i \in I} A_{i})
\]

b) What is the relationship between \( \bigcup_{i \in I} \wp(A_{i}) \) and \( \wp(\bigcup_{i \in I} A_{i}) \)? (4 pts.)

**Answer:** We have \( \bigcup_{i \in I} \wp(A_{i}) \subseteq \wp(\bigcup_{i \in I} A_{i}) \). Indeed,

\[
X \in \bigcup_{i \in I} \wp(A_{i}) \quad \text{if and only if} \quad X \in \wp(A_{i}) \text{ for some } i \in I \\
\text{if and only if} \quad X \subseteq A_{i} \text{ for some } i \in I \\
\text{implies} \quad X \subseteq \bigcup_{i \in I} A_{i} \\
\text{if and only if} \quad X \in \wp(\bigcup_{i \in I} A_{i})
\]

The reverse inclusion is false, in fact if one of the sets \( A_{i} \) does not contain all the others, then \( \bigcup_{i \in I} A_{i} \notin \wp(\bigcup_{i \in I} A_{i}) \).

4. What is the set \( (X \times Y) \cap (Y \times X) \)? (3 pts.)

**Answer:** Let \( (u, v) \in (X \times Y) \cap (Y \times X) \). Since \( (u, v) \in X \times Y \), \( u \) is an element of \( X \) and \( v \) is an element of \( Y \). Similarly, since \( (u, v) \in Y \times X \), \( u \) is an element of \( Y \) and \( v \) is an element of \( X \). Thus both \( u \) and \( v \) are elements of \( X \cap Y \), i.e. \( (u, v) \in (X \cap Y) \times (X \cap Y) \).

The reverse inclusion \( (X \cap Y) \times (X \cap Y) \subseteq (X \times Y) \cap (Y \times X) \) is trivial.

5. Let \( \alpha \) be a set such that \( x \subseteq \alpha \) for all \( x \in \alpha \). Show that \( \alpha \cup \{ \alpha \} \) has the same property. Give four examples of such sets. (4 pts.)

**Proof:** Let \( x \in \alpha \cup \{ \alpha \} \). Then either \( x \in \alpha \) or \( x \in \{ \alpha \} \).

If \( x \in \alpha \), then by assumption \( x \subseteq \alpha \). Since \( \alpha \subseteq \alpha \cup \{ \alpha \} \), in this case we get \( x \subseteq \alpha \cup \{ \alpha \} \).

If \( x \in \{ \alpha \} \), then \( x = \alpha \), and so \( x = \alpha \subseteq \alpha \cup \{ \alpha \} \).

Since \( \emptyset \) has the property stated, starting from \( \emptyset \), we can get as many examples as we wish to, here are the first four:

\[
\emptyset = 0 \\
0 \cup \{0\} = 1 \\
1 \cup \{1\} = 2 \\
2 \cup \{2\} = 3
\]
6. Let $\Gamma$ be a graph such that for any vertices $\alpha, \alpha_1, \beta, \beta_1$, if $\alpha \neq \alpha_1$ and $\beta \neq \beta_1$, then there is a $\phi \in \text{Aut}(\Gamma)$ such that $\phi(\alpha) = \beta$ and $\phi(\alpha_1) = \beta_1$.

What can you say about $\Gamma$? (5 pts.)

Answer: Then either the graph is the complete graph (all possible edges exist) or the graph without edges at all. Indeed, otherwise we may find $\alpha, \alpha_1$ and $\beta, \beta_1$ such that $\alpha$ and $\alpha_1$ are connected (hence $\alpha \neq \alpha_1$) and $\beta$ and $\beta_1$ are not connected, but then it is impossible to send the connected pair $(\alpha, \alpha_1)$ to the nonconnected pair $(\beta, \beta_1)$.

7. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a one to one map such that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(x^2) = \phi(x)^2$ for all $x, y \in \mathbb{R}$. Show that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$ and $\phi(q) = q$ for all $q \in \mathbb{Q}$. (10 pts.)

Proof: For any $x, y \in \mathbb{R}$, we have $\phi(x)^2 + 2\phi(x)\phi(y) + \phi(y)^2 = (\phi(x) + \phi(y))^2 = \phi(x + y)^2 = \phi((x + y)^2) = \phi(x^2 + 2xy + y^2) = \phi(x^2) + 2\phi(xy) + \phi(y^2) = \phi(x)^2 + 2\phi(xy) + \phi(y)^2$ and so $2\phi(x)\phi(y) = 2\phi(xy)$ and simplifying, we get $\phi(x)\phi(y) = \phi(xy)$. This proves the first part.

Since $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$, we must have $\phi(0) = 0$.

Since $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$, we must have $\phi(1) = 0$ or $\phi(1) = 1$. But the first case is forbidden because $\phi$ is one to one and $\phi(0) = 0$ already. Hence $\phi(1) = 1$.

Now, it follows easily by induction that $\phi(n) = n$ for all $n \in \mathbb{N}$ because $\phi(n + 1) = \phi(n) + \phi(1) = \phi(n) + 1 = n + 1$ (the last equality is the inductive hypothesis).

Also, for $n \in \mathbb{N}$, we have $0 = \phi(0) = \phi(n + (-n)) = \phi(n) + \phi(-n)$ and so $\phi(-n) = -\phi(n) = -n$. Thus $\phi(n) = n$ for all $n \in \mathbb{Z}$.

Now if $q \in \mathbb{Q}$, then $q = n/m$ some $n, m \in \mathbb{Z}$ and $m \neq 0$. Then we have $n = \phi(n) = \phi(n/m) = \phi(m)\phi(n/m) = m\phi(n/m)$ and so $\phi(n/m) = n/m$, i.e. $\phi(q) = q$.

8. Given a set $X$, define $\varphi^n(X)$ as follows by induction on $n$: $\varphi^0(X) = X$ and $\varphi^{n+1}(X) = \varphi(\varphi^n(X))$.

a) Is there a natural number $n$ such that for any set $X$, $\{\emptyset\}, \{\{X\}\} \in \varphi^n(X)$? (8 pts.)

Answer: For $n \geq 4$ note the equivalence of the following propositions:

- $\{\emptyset\}, \{\{X\}\} \in \varphi^n(X)$
- $\{\emptyset\}, \{\{X\}\} \subseteq \varphi^{n-1}(X)$
- $\emptyset, \{X\} \in \varphi^{n-1}(X)$
- $\emptyset, \{X\} \subseteq \varphi^{n-2}(X)$
- $\emptyset, \{X\} \in \varphi^{n-2}(X)$
- $\emptyset, \{X\} \subseteq \varphi^{n-3}(X)$
- $\emptyset, \{X\} \subseteq \varphi^{n-3}(X)$
- $X \in \varphi^{n-3}(X)$
- $X \subseteq \varphi^{n-4}(X)$

$\{\emptyset\}, \{\{X\}\} \notin \varphi^n(X)$
If \( n = 4 \), the last condition holds for all \( X \).

Does it hold for \( n = 5 \), i.e. do we have \( X \subseteq \wp(X) \) for all \( X \) ? For this condition to hold, we need any element of \( X \) to be a subset of \( X \), and this does not always hold. For any \( i \geq 1 \), one can find a set \( X \) such that \( X \not\subseteq \wp^i(X) \). Thus the condition does not hold for any \( n \geq 5 \) (details are left as an exercise).

For \( n = 0, 1, 2, 3 \), find examples of \( X \) such that \( \{\emptyset\}, \{\{X\}\} \not\in \wp^n(X) \).

b) Show that \( \wp(\wp^n(X)) = \wp^n(\wp(X)) \) for all sets \( X \) and all natural numbers \( n \). (8 pts.)

**Proof:** We proceed by induction on \( n \). The condition certainly holds for \( n = 0 \). Assume it holds for \( n \). We have \( \wp(\wp^{n+1}(X)) = \wp(\wp^n(\wp(X))) = \wp^{n+1}(\wp(X)) \).

c) Show that \( \wp^n(\wp^m(X)) = \wp^m(\wp^n(X)) \) for all sets \( X \) and all natural numbers \( n \) and \( m \). (8 pts.)

**Proof:** We proceed by induction on \( m \). The condition certainly holds for \( m = 0 \). By part (b) it also holds for \( m = 1 \). Assume it holds for \( m \). We have \( \wp^n(\wp^{m+1}(X)) = \wp^n(\wp(\wp^m(X))) = \wp^n(\wp(X)) = \wp^{n+1}(\wp^m(X)) \).

9. Define a partial order \( \prec \) on \( \mathbb{N} \setminus \{0, 1\} \) by \( x \prec y \) if and only if \( x^2 \mid y \). Describe all the automorphisms of this poset. (5 pts.)

**Answer:** The minimal elements of this ordered set (call it \( \Gamma \)) are the square free numbers. Thus any automorphism of \( \Gamma \) should send the square free numbers onto the square free numbers. But the prime numbers have a privilege. Indeed if \( p \) is a prime number, then \( p \) has an immediate successor (namely \( p^2 \)) that has only one predecessor, namely \( p \). Thus any automorphism should be multiplicative and be given by a permutation of primes.

10. Let \( X \) be a set. Let \( \Gamma \) be the set of subsets of \( X \) with two elements. On \( \Gamma \) define the relation \( \alpha R \beta \) if and only if \( \alpha \cap \beta = \emptyset \). Then \( \Gamma \) becomes a graph with this relation.
a) Calculate \( \text{Aut}(\Gamma) \) when \( |X| = 4 \). (3 pts.)

**Answer:** The graph \( \Gamma \) is just six vertices joined two by two. A group isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^3 \) preserves the edges. And \( \text{Sym}(3) \) permutes the edges. Thus the group has 8 \( \times \) 3! = 48 elements.

More formally, one can prove this as follows. Let the points be \( \{1, 2, 3, 4, 5, 6\} \) and the edges be \( v_1 = (1, 4) \), \( v_2 = (2, 5) \) and \( v_3 = (3, 6) \). We can embed \( \text{Sym}(3) \) in \( \text{Aut}(\Gamma) \leq \text{Sym}(6) \) via

\[
\begin{align*}
\text{Id}_3 &\quad \leftrightarrow \quad \text{Id}_6 \\
(12) &\quad \leftrightarrow \quad (12)(45) \\
(13) &\quad \leftrightarrow \quad (13)(46) \\
(23) &\quad \leftrightarrow \quad (23)(56) \\
(123) &\quad \leftrightarrow \quad (123)(456) \\
(132) &\quad \leftrightarrow \quad (132)(465)
\end{align*}
\]
For any $\phi \in \text{Aut}(\Gamma)$ there is an element $\alpha$ in the image of $\text{Sym}(3)$ such that $\alpha^{-1}\phi$ preserves the three edges $v_1 = (1,4), v_2 = (2,5)$ and $v_3 = (3,6)$. Thus $\alpha^{-1}\phi \in \text{Sym}(1,4) \times \text{Sym}(2,5) \times \text{Sym}(3,6) \simeq (\mathbb{Z}/2\mathbb{Z})^3$. It follows that $\text{Aut}(\Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \text{Sym}(3)$ (to be explained next year).

b) Draw the graph $\Gamma$ when $X = \{1, 2, 3, 4, 5\}$. (3 pts.)

There are ten points. Draw two pentagons one inside the other. Label the outside points as $\{1, 2\}, \{3, 4\}, \{5, 1\}, \{2, 3\}, \{4, 5\}$. Complete the graph.

c) Show that $\text{Sym}(5)$ imbeds in $\text{Aut}(\Gamma)$ naturally. (You have to show that each element $\sigma$ of $\text{Sym}(5)$ gives rise to an automorphism $\tilde{\sigma}$ of $\Gamma$ in such a way that the map $\sigma \mapsto \tilde{\sigma}$ is an injection from $\text{Sym}(5)$ into $\text{Aut}(\Gamma)$ and that $\tilde{\sigma_1} \circ \tilde{\sigma_2} = \tilde{\sigma_1 \circ \sigma_2}$). (8 pts.)

d) Show that $\text{Aut}(\Gamma) \simeq \text{Sym}(5)$. (12 pts.)

**Proof of (c) and (d):** Clearly any element of $\sigma \in \text{Sym}(5)$ gives rise to an automorphism $\tilde{\sigma}$ of $\Gamma$ via $\tilde{\sigma}\{a, b\} = \{\sigma(a), \sigma(b)\}$. The fact that this map preserves the incidence relation is clear. This map is one to one because if $\tilde{\sigma} = \tilde{\tau}$, then for all distinct $a, b, c$, we have $\{\sigma(b)\} = \{\sigma(a), \sigma(b)\} \cap \{\sigma(b), \sigma(c)\} = \tilde{\sigma}\{a, b\} \cap \tilde{\sigma}\{b, c\} = \tilde{\tau}\{a, b\} \cap \tilde{\tau}\{b, c\} = \{\tau(a), \tau(b)\} \cap \{\tau(b), \tau(c)\} = \{\tau(b)\}$ and hence $\sigma(b) = \tau(b)$.

Let $\phi \in \text{Aut}(\Gamma)$. We will compose $\phi$ by elements of $\text{Sym}(5)$ to obtain the identity map. There is an $\sigma \in \text{Sym}(5)$ such that $\phi\{1, 2\} = \tilde{\sigma}\{1, 2\}$ and $\phi\{3, 4\} = \tilde{\sigma}\{3, 4\}$. Thus, replacing $\phi$ by $\sigma^{-1}\phi$, we may assume that $\phi$ fixes the vertices $\{1, 2\}$ and $\{3, 4\}$. Now $\phi$ must preserve or exchange the vertices $\{3, 5\}$ and $\{4, 5\}$. By applying the element $(34)$ of $\text{Sym}(5)$ we may assume that these two vertices are fixed as well. Now $\phi$ must preserve or exchange the vertices $\{1, 3\}$ and $\{2, 3\}$. By applying the element $(12)$ of $\text{Sym}(5)$ we may assume that these two vertices are fixed as well. Now all the vertices must be fixed.