Math 111

Resit (1/2)

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A set X together with a function $d : X \times X \to \mathbb{R}^{\geq 0}$ satisfying

D1. For all $x, y \in X$, $d(x, y) = 0$ iff $x = y$.

D2. For all $x, y \in X$, $d(x, y) = d(y, x)$.

D3. For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$. **(Triangular Inequality)**

is called a **metric space**. The function d is then called a **metric** on X.

Let (X, d) be a metric space. For $a \in X$ and $r \in \mathbf{R}$, we define,

$$B(a, r) = \{x \in X : d(a, x) = 0\}.$$

$B(a, r)$ is called the **open ball** of **center** a and **radius** r.

A subset U of a metric space (X, d) is called **open** if it is a union of arbitrarily many open balls (of various centers and various radii).

1. Let $X = \mathbb{R}^n$ and $d_1 : X \times X \to \mathbb{R}^{\geq 0}$ be given by

$$d_1((x_1, ..., x_n), (y_1, ..., y_n)) = |x_1 - y_1| + ... + |x_n - y_n|.$$

1a. *Show that* (X, d_1) *is a metric space.*

Answer: We have to check D1, D2 and D3.

D1. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. By definition of d_1, $d_1(x, y) = 0$ iff $|x_1 - y_1| + ... + |x_n - y_n| = 0$. But each of the summands $|x_i - y_i|$ is nonnegative. Therefore the sum $|x_1 - y_1| + ... + |x_n - y_n|$ is zero iff each summand $|x_i - y_i|$ is zero, i.e. iff $x_i = y_i$ for all $i = 1, ..., n$, i.e. iff $x = y$.

D2. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. Then

$$\begin{aligned} d_1(x, y) &= d_1((x_1, ..., x_n), (y_1, ..., y_n)) \\ &= |x_1 - y_1| + ... + |x_n - y_n| \\ &= |y_1 - x_1| + ... + |y_n - x_n| \\ &= d_1((y_1, ..., y_n), (x_1, ..., x_n)) \\ &= d_1(y, x). \end{aligned}$$

D3. We will use the inequality $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbf{R}$.

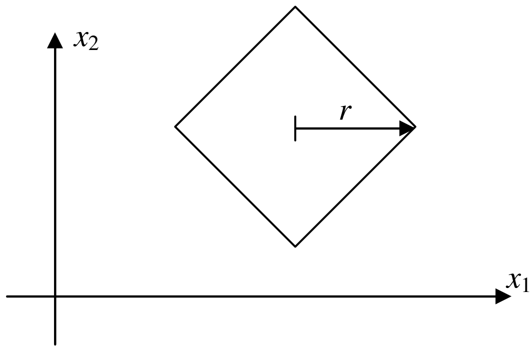
Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ be any three elements of X. We compute:

$$\begin{aligned} d_1(x, y) &= |x_1 - y_1| + ... + |x_n - y_n| \\ &= |(x_1 - z_1) + (z_1 - y_1)| + ... + |(x_n - z_n) + (z_n - y_n)| \\ &\leq (|x_1 - z_1| + |z_1 - y_1|) + ... + (|x_n - z_n| + |z_n - y_n|) \\ &= (|x_1 - z_1| + ... + |x_n - z_n|) + (|z_1 - y_1| + ... + |z_n - y_n|) \\ &= d_1(x, z) + d_1(z, y). \end{aligned}$$

Thus d_1 is a metric.

1b. *Take* $n = 2$. *Given* $a = (a_1, a_2)$, *find geometrically* $B(a, r)$. *Draw it.*

Answer: $B(a, r) = \{x = (x_1, x_2) \in \mathbf{R}^2 : d_1(x, a) < r\} = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1 - a_1| + |x_2 - a_2| < r\}$. Geometrically, this is the interior of a diamond of center (a_1, a_2) as drawn below:



Thus the ball in this metric space is not the "usual" ball.

2. Let $X = \mathbb{R}^n$ and $d_2 : X \times X \to \mathbb{R}^{\geq 0}$ be given by

$$d_2((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

2a. *Show that* (X, d_2) *is a metric space.*

Answer: We have to check D1, D2 and D3.

D1. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. By definition of d_2, $d_2(x, y) = 0$ iff $\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} = 0$, i.e. iff $(x_1 - y_1)^2 + ... + (x_n - y_n)^2 = 0$. But each of the summands $(x_i - y_i)^2$ is nonnegative. Therefore the sum $(x_1 - y_1)^2 + ... + (x_n - y_n)^2$ is zero iff each summand $(x_i - y_i)^2$ is zero, i.e. iff $x_i = y_i$ for all $i = 1, ..., n$, i.e. iff $x = y$.

D2. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. Then

$$\begin{aligned} \mathrm{d}_2(x, y) &= d_2((x_1, ..., x_n), (y_1, ..., y_n)) \\ &= \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} \\ &= \sqrt{(y_1 - x_1)^2 + ... + (y_n - x_n)^2} \\ &= d_2((y_1, ..., y_n), (x_1, ..., x_n)) \\ &= d_2(y, x). \end{aligned}$$

D3. Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ be any three elements of X. We have to show that $d_2(x, y) \leq d_2(x, z) + d_2(z, y)$, i.e. that

$$\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} \leq \sqrt{(x_1 - z_1)^2 + ... + (x_n - z_n)^2} + \sqrt{(z_1 - y_1)^2 + ... + (z_n - y_n)^2}\ .$$

Note that when $n = 1$, this is just the above question, in other words $d_1 = d_2$ when $n = 1$. When $n = 2$, this is the well known triangular inequality for the Euclidean plane.

Setting $a_i = x_i - z_i$ and $b_i = z_i - y_i$, the above inequality that we want to show is equivalent to the following one:

$$\sqrt{(a_1 + b_1)^2 + ... + (a_n + b_n)^2} \leq \sqrt{a_1^2 + ... + a_n^2} + \sqrt{b_1^2 + ... + b_n^2}\ .$$

Since both sides are positive, squaring them, we see that we have to show the inequality

$$(a_1 + b_1)^2 + ... + (a_n + b_n)^2 \leq a_1^2 + ... + a_n^2 + 2\sqrt{a_1^2 + ... + a_n^2}\ \sqrt{b_1^2 + ... + b_n^2} + b_1^2 + ... + b_n^2.$$

Computing the squares on the left hand side and then simplifying, we get the following equivalent inequality:

$$a_1b_1 + ... + a_nb_n \leq \sqrt{a_1^2 + ... + a_n^2}\ \sqrt{b_1^2 + ... + b_n^2}$$

Note that we may assume that all the a_i and b_i are positive, in other words if the inequality holds for positive numbers, then it also holds for negative numbers, thus it is enough to show this inequality for positive real numbers. Therefore, squaring both sides we get the following equivalent inequality:

$$\sum_{i,j} a_i b_i a_j b_j \leq \sum_{i,j} a_i^2 b_j^2$$

Simplifying the terms of the form $a_i^2 b_i^2$, we get the following equivalent inequality:

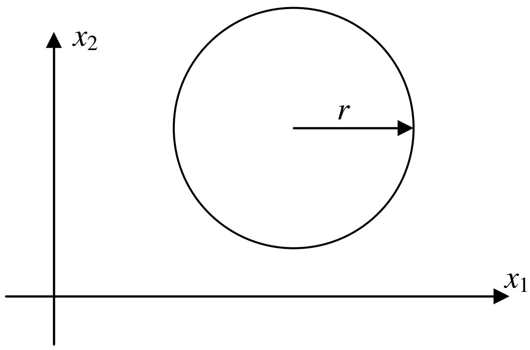
$$\sum_{i \neq j} a_i b_i a_j b_j \leq \sum_{i \neq j} a_i^2 b_j^2 .$$

Taking the left hand side to the right, we get the next equivalent inequality:

$$\sum_{i \neq j} a_i^2 b_j^2 - 2\sum_{i \prec j} a_i b_j a_j b_i \geq 0 .$$

The left hand side is nothing else than the sum of the squares $(a_i b_j - a_j b_i)^2$ for $i \neq j$, which is therefore positive. This proves the triangular inequality.

2b. Take $n = 2$. Given $a = (a_1, a_2)$, find geometrically $B(a, r)$. Draw it.
This is the standard open ball, the inside of the standard circle of center a and radius r.



3. Let $X = \mathbb{R}^n$ and $d_3 : X \times X \to \mathbb{R}^{\geq 0}$ be given by

$$d_3((x_1, ..., x_n), (y_1, ..., y_n)) = \max(|x_1 - y_1|, ..., |x_n - y_n|)$$

3a. *Show that* (X, d_3) *is a metric space.*

Answer: We have to check D1, D2 and D3.

D1. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. By definition of d_3, $d_3(x, y) = 0$ iff $\max(|x_1 - y_1|, ..., |x_n - y_n|) = 0$. But each of the terms $|x_i - y_i|$ is nonnegative. Therefore $\sup(|x_1 - y_1|, ..., |x_n - y_n|)$ is zero iff each term $|x_i - y_i|$ is zero, i.e. iff $x_i = y_i$ for all $i = 1, ..., n$, i.e. iff $x = y$.

D2. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be any two elements of X. Then

$$\begin{aligned} d_3(x, y) &= d_3((x_1, ..., x_n), (y_1, ..., y_n)) \\ &= \max(|x_1 - y_1|, ..., |x_n - y_n|) \\ &= \max(|y_1 - x_1|, ..., |y_n - x_n|) \\ &= d_3((y_1, ..., y_n), (x_1, ..., x_n)) \\ &= d_3(y, x). \end{aligned}$$

D3. We will use the inequality $|a - b| \leq |a - c| + |c - b|$ for all $a, b, c \in \mathbb{R}$.

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ be any three elements of X. We compute:

$$\begin{aligned} d_3(x, y) &= \max(|x_1 - y_1|, ..., |x_n - y_n|) \\ &= \max(|(x_1 - z_1) + (z_1 - y_1)|, ..., |(x_n - z_n) + (z_n - y_n)|) \\ &\leq \max(|x_1 - z_1| + |z_1 - y_1|, ..., |x_n - z_n| + |z_n - y_n|) \end{aligned}$$

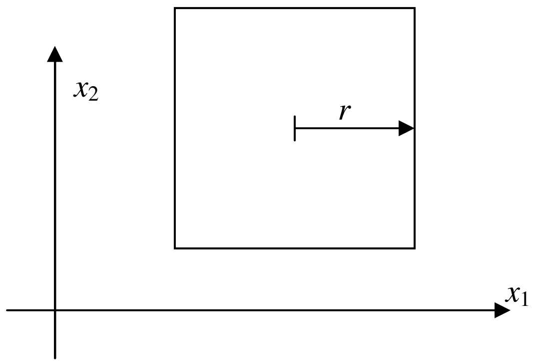
$\leq \max(|x_1 - z_1|, \ldots, |x_n - z_n|) + \max(|z_1 - y_1|, \ldots, |z_n - y_n|)$
$= d_3(x, z) + d_1(z, y)$.

Thus d_3 is a metric.

3b. *Take $n = 2$. Given $a = (a_1, a_2)$, find geometrically $B(a, r)$. Draw it.*

Answer: $B(a, r) = \{x = (x_1, x_2) \in \mathbb{R}^2 : d_3(x, a) < r\}$
$= \{(x_1, x_2) \in \mathbb{R}^2 : \max(|x_1 - a_1|, |x_2 - a_2|) < r\}$
$= \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| < r \text{ and } |x_2 - a_2| < r\}$.

Geometrically, this is the interior of the square of center (a_1, a_2) and of sides of length $2r$ as drawn below:



Thus the ball in this metric space is not the "usual" ball.

4. On a set X define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

4a. *Show that (X, d) is a metric space.*

Answer: We have to check D1, D2 and D3.

D1. Let x and y be any two elements of X. By definition of d, $d(x, y) = 0$ iff $x = y$ (otherwise the distance is 1).

D2. Let x and y be any two elements of X. Then $d(x, y) = d(y, x)$ by definition of d.

D3. We will show $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y and z. This is clear if any two of x, y, z are equal to each other. Assume they are all different. Then $d(x, y) = 1 \leq 2 = 1 + 1 = d(x, z) + d(z, y)$.

4b. *What are the open subsets of X?*

If $x \in X$, then $B(x, 1/2) = \{x\}$. Thus singleton sets are open balls. It follows that every subset of X is open.

5c. *Show that a subset U of a metric space (X, d) is open iff for any $a \in U$, there is an $\varepsilon > 0$ (that may depend on a) such that $B(a, \varepsilon) \subseteq U$.*

(⇒) Let U be an open subset of the metric space X. Let $a \in U$. Since U is a union of open balls, a is in an open ball $B(b, r)$ which is a subset of U. Let $\varepsilon = r - d(a, b)$. Since $a \in B(b, r)$, $d(a, b) < r$, so that $\varepsilon > 0$. We claim that $B(a, \varepsilon) \subseteq B(b, r)$. The latter ball being in U, this will prove our asserion. Let $x \in B(a, \varepsilon)$. Thus $d(x, a) < \varepsilon$. Now $d(b, x) = d(x, b) \leq d(x, a) + d(a, b) < \varepsilon + d(a, b) = r$. Hence $d(b, x) < r$ and $x \in B(b, r)$.

(⇐) For each $a \in U$ choose an $\varepsilon_a > 0$, such that $B(a, \varepsilon_a) \subseteq U$. Since $a \in B(a, \varepsilon_a)$, the union of these balls (a varying in U) is U.

6. *Show that the metric spaces* ($\mathbb{R}^2$, d_1), ($\mathbb{R}^2$, d_2) *and* ($\mathbb{R}^2$, d_3), *as defined above, have the same open subsets.* (Hint: Use #5).

It is clear from the picctures drawn in Questions #1, 2 and 3 that if B is a ball with respect to one of the three metrics and if $a \in B$, then there are balls with respect to the other metrics that are centered at a that are contained in B. Hence, by the previous question, B is open with respect to the other two metrics. It follows that every open subset with respect to this metric is an open subset with respect to the other metrics.

7a. *Show that union of arbitrarily many open subsets of a metric space is open.*

Clear.

7b. *Show that the intersection of two open balls of a metric space is open.*

Let $B(a, r)$ and $B(b, s)$ be two open balls. Let $c \in B(a, r) \cap B(b, s)$. We will show that for some $\varepsilon > 0$, $B(c, \varepsilon) \subseteq B(a, r) \cap B(b, s)$. This will prove our assertion by Question #5. Let $\varepsilon = \min(r - d(a, c), s - d(b, c))$. Exactly as in the first part of Question 5, $B(c, \varepsilon) \subseteq B(a, r) \cap B(b, s)$.

7c. *Show that the intersection of two open subsets of a metric space is open.*

Let U and V be two open subsets. Let U be the union of open balls $(B_i)_{i\in I}$ and V be the union of open balls $(C_j)_{j\in J}$. Then $U \cap V$ is the union of $(B_i \cap C_j)_{i,j}$. But each $B_i \cap C_j$ is open by 7b. Thus $U \cap V$ is open.

8. Let (X, d) be a metric space. Let $(x_n)_n$ be a sequence in X. We say that $(x_n)_n$ is a **Cauchy sequence** if for all $\varepsilon > 0$ there is a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $n, m > N$. Let $x \in X$. We say that x is a **limit** of the sequence $(x_n)_n$ if for all $\varepsilon > 0$ there is an N such that for all $n > N$, $d(x_n, x) < \varepsilon$; we then say that the sequence $(x_n)_n$ is convergent.

8a. *Show that the limit of a convergent sequence is unique.*

Let $(x_n)_n$ be a convergent seqence. Assume it converges to a and b. We will show that $a = b$.

First Proof: Assume not. Then $d(a, b) > 0$. Let $0 < \varepsilon < d(a, b)$ be fixed. Let N_1 be such that if $n > N_1$ then $d(x_n, a) < \varepsilon/2$. Let N_2 be such that if $n > N_2$ then $d(x_n, b) < \varepsilon/2$. Now let $n > \max(N_1, N_2)$. Then $d(a, b) \leq d(a, x_n) + d(x_n, b) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, contradicting the choice of ε.

Second Proof: Let $\varepsilon > 0$. Let N_1 be such that if $n > N_1$ then $d(x_n, a) < \varepsilon/2$. Let N_2 be such that if $n > N_2$ then $d(x_n, b) < \varepsilon/2$. Now let $n > \max(N_1, N_2)$. Then $d(a, b) \leq d(a, x_n) + d(x_n, b) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since this is true for all $\varepsilon > 0$, we get $d(a, b) = 0$.

8b. *Show that every convergent sequence is Cauchy.*

Let $(x_n)_n$ be a seqence converging to a. Let $\varepsilon > 0$. Let N be such that if $n > N$ then $d(x_n, a) < \varepsilon/2$. Now for $n, m > N$, $d(x_n, x_m) \leq d(x_n, a) + d(a, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $(x_n)_n$ is a Cauchy sequence.

8c. *Show that a constant sequence is convergent.*

Let $(x_n)_n$ be a constant seqence. Thus $x_n = a$ for all n. We claim that a is the limit of the sequence $(x_n)_n$. Let $\varepsilon > 0$. Let $N = 1$. For any $n > N$, $d(x_n, a) = d(a, a) = 0 < \varepsilon$. This proves that a is the limit of the constant sequence.

8d. *Suppose X is finite. Show that every Cauchy sequence in X is convergent.*

Let $(x_n)_n$ be a Cauchy seqence. Let $\varepsilon = \min\{d(x, y) : x, y \in X \text{ and } x \neq y\}$. Let N be such that for all $n, m > N$, $d(x_n, x_m) < \varepsilon$. Thus $x_n = x_m$ for all $n, m > N$. Let a be this value. Thus $x_n = a$ for all $n > N$. One can show as in Question 8c that a is the limit of this sequence.