Math 111

Third Midterm (Ordinals)
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Ali Nesin

To solve a question, you may assume you have solved all the previous questions. Take your time. Do not try to cheat, because 1) this is immoral, 2) you won't succeed. Do not lie to yourself, be honest, only explain what you understood. Do not try to succeed at any rate, at any price. Write clearly, with brieve and correct English sentences. Try to avoid symbols like $\forall$, $\exists$, $\Rightarrow$.

You may need to assume that no set is its own element.

Let $(X, <)$ be a totally ordered set. We say that $(X, <)$ is a **well-ordered** set (or that $<$ well-orders X) if every nonempty subset of X contains a minimal element (a least) for that order, i.e., if for every nonempty subset A of X, there is an $m \in A$ such that $m \leq a$ for all a in A. Clearly, given A, such an m is unique. In particular if $X \neq \emptyset$, X has a least element.

1. Give two different examples of well-ordered sets one of which is infinite. (3 pts.)

2. Show that every subset of a well-ordered set is a well-ordered set with respect to the induced order. (3 pts.)

3. Show that the minimal element m of a nonempty subset A of a totally ordered set $(X, <)$ is unique. (4 pts.)

4. Is it true that that the union of an increasing chain of well-ordered sets is well-ordered? (5 pts.)

If X is a set, we set $X^+ = X \cup \{X\}$.

5. Assume X is a well-ordered set and that $X \notin X$. Order X^+ by extending the order of X and by stating that X is larger than its elements (i.e. to get X^+, put the element X to "the very end" of X). Show that X^+ is also a well-ordered set. (5 pts.)

6. If $(X, <)$ is an ordered set and $x \in X$, we define the **initial segment** of x as

$$s(x) = \{y \in X : y < x\}.$$

What is $s(o)$ if o is the minimal element of X? (2 pts.)

7. Let X and Y be two well-ordered sets. Let

$$A = (X \times \{0\}) \cup (Y \times \{1\}).$$

Order A as follows: For all $x, x_1, x_2 \in X$ and for all $y, y_1, y_2 \in Y$

$(x_1, 0) < (x_2, 0)$ iff $x_1 < x_2$.

$(y_1, 1) < (y_2, 1)$ iff $y_1 < y_2$.

$(x, 0) < (y, 1)$.

Show that the above relation well-orders A. (6 pts.)

8. (Transfinite Induction) Let $(X, <)$ be a well-ordered set and let $A \subseteq X$ be such that for all $x \in X$, if $s(x) \subseteq A$, then $x \in A$. Show that $A = X$. (15 pts.)

An **ordinal** is a well-ordered set α such that $\beta = s(\beta)$ for all $\beta \in \alpha$. Thus an ordinal is a set α well-ordered by the relation $\in$, i.e. the binary relation $<$ on α defined by "$\beta < \gamma$ iff $\beta \in \gamma$" well-orders α.

9. Show that $\varnothing$ is an ordinal. (2 pts.)

10. Show that if $\alpha \neq \varnothing$ is an ordinal, then $\varnothing \in \alpha$ and $\varnothing$ is the least element of α. (4 pts.)

11. Show that if an ordinal α has at leat two elements, then $1 \in \alpha$. (5 pts.)

12. Show that there is a unique ordinal α with exactly 3 elements. (5 pts.)

13. Show that if α is an ordinal and $\beta \in \alpha$, then $\beta \subseteq \alpha$. (5 pts.)

14. Show that every element of an ordinal is an ordinal. (5 pts.)

15. Show that if α is an ordinal, then α^+ is also an ordinal. (5 pts.)

16. Let α be an ordinal and $\beta \in \alpha$. Show that either $\beta^+ \in \alpha$ or $\beta^+ = \alpha$. (15 pts.)

17. Let α and β be ordinals. Assume that $\beta \subset \alpha$. Show that $\beta \in \alpha$. (15 pts.)