Set Theory

Midterm
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1. Define the function $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) = 1/q$ if $x \in \mathbb{Q}$, $x = p/q$ where $p, q \in \mathbb{Z}$, $(p, q) = 1$ and $q > 0$. Show that f is continuous only at $a \notin \mathbb{R} \setminus \mathbb{Q}$.

2. We say that two partially ordered sets $(X, <_X)$ and $(Y, <_Y)$ are isomorphic if there is a bijection $f : X \to Y$ such that for all $x, x' \in X$,

$$x <_X x' \text{ iff } f(x) <_Y f(x').$$

How many nonisomorphic partial orders on a set X with 1, 2, 3, 4 and 5 points.

3. Let X be a set. A **filter** on X is a set $\mathfrak{I}$ of subsets of X that satisfies the following properties:

i) If $A \in \mathfrak{I}$ and $A \subseteq B \subseteq X$, then $B \in \mathfrak{I}$.
ii) If A and B are in $\mathfrak{I}$, then so is $A \cap B$.
iii) $\varnothing \notin \mathfrak{I}$ and $X \in \mathfrak{I}$.

If $A \subseteq X$ is a fixed nonempty subset of X, then the set of subsets of X that contain A is a filter on X, called **principal filter**. We will denote this filter by $\mathfrak{I}(A)$.

If X is infinite, then the set of cofinite subsets of X is a filter, called the **Fréchet filter**.

3a. Show that the Fréchet filter is nonprincipal.

A filter is called **ultrafilter** if it is a maximal filter.

3b. Show that a principle filter $\mathfrak{I}(A)$ is an ultrafilter iff A is a singleton set.

3c. Show that a filter $\mathfrak{I}$ is an ultrafilter iff for all $A \subseteq X$, either A or A^c is in $\mathfrak{I}$.

3d. Show that every nonprinciple ultrafilter contains the Fréchet filter.

3e. Show that if X is infinite then there are nonprincipal ultrafilters on X. (Hint: Use Zorn's Lemma).