1. Define the function $f: \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) = 1/q$ if $x \in \mathbb{Q}$, $x = p/q$ where $p, q \in \mathbb{Z}$, $(p, q) = 1$ and $q > 0$. Show that $f$ is continuous only at $a \notin \mathbb{R} \setminus \mathbb{Q}$.

2. We say that two partially ordered sets $(X, <_X)$ and $(Y, <_Y)$ are isomorphic if there is a bijection $f: X \to Y$ such that for all $x, x' \in X$,

$$x <_X x' \iff f(x) <_Y f(x').$$

How many nonisomorphic partial orders on a set $X$ with 1, 2, 3, 4 and 5 points.

3. Let $X$ be a set. A filter on $X$ is a set $\mathcal{F}$ of subsets of $X$ that satisfies the following properties:

   i) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
   ii) If $A$ and $B$ are in $\mathcal{F}$, then so is $A \cap B$.
   iii) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

If $A \subseteq X$ is a fixed nonempty subset of $X$, then the set of subsets of $X$ that contain $A$ is a filter on $X$, called principal filter. We will denote this filter by $\mathcal{F}(A)$.

If $X$ is infinite, then the set of cofinite subsets of $X$ is a filter, called the Fréchet filter.

3a. Show that the Fréchet filter is nonprincipal.

A filter is called ultrafilter if it is a maximal filter.

3b. Show that a principle filter $\mathcal{F}(A)$ is an ultrafilter iff $A$ is a singleton set.

3c. Show that a filter $\mathcal{F}$ is an ultrafilter iff for all $A \subseteq X$, either $A$ or $A^c$ is in $\mathcal{F}$.

3d. Show that every nonprinciple ultrafilter contains the Fréchet filter.

3e. Show that if $X$ is infinite then there are nonprincipal ultrafilters on $X$. (Hint: Use Zorn’s Lemma).