Math 111 (Set Theory)

Midterm 1/3

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Advise: Unjustified answers will not be accepted. Make full sentences with correct punctuation. Write concisely and clearly with short sentences. Do not exaggerate the use of mathematical symbols such as $\forall$, $\exists$, $\Rightarrow$, and never use them in English sentences. No answer will receive 0 points, unlike complete nonsense that will not even be graded. In other words, do not try to hunt for points by writing anything that crosses your mind. Be honnest with yourself. Stay in class and think till the last minute of the exam. Do not try to remember; think again, rediscover. Do not worry about your grade. Do not try to cheat, because, first, I will understand it, and second, a good mathematician never cheats or lies!

Finally, enjoy the exam.

1. Show that $\varnothing$ is a subset of every set. (4 pts.)

2. Let X be any set. Show that there is no map from X **onto** the set $\wp(X)$ of subsets of X. (8 pts.)

3. Let X be a set. Show that there is a one-to-one correspondance (i.e. a bijection) between $\wp(X)$ and the set ${}^{X}2$ of functions from X into the set $\{0, 1\}$. (8 pts.)

4. Let X and Y be two sets and f and g be two functions from X into Y. Let A and B be two subsets of X such that $X = A \sqcup B$. Show that the rule

$$h(x) = \begin{cases} f(x) & if\ x \in A \\ g(x) & if\ x \in B \end{cases}$$

defines a function from X into Y. (8 pts.)

5. In this exercise we assume the knowledge of high school mathematics. Let $\mathbb{R}$ denote the set of real numbers. A subset C of $\mathbb{R}^n$ is called **convex** if for every $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in C and for every real number t in the interval $[0, 1]$, the point

$$(tx_1 + (1 - t)y_1, ..., tx_n + (1 - t)y_n)$$

is also in C.

5a. For $a \leq b$ two real numbers and $t \in [0, 1]$, show that $a \leq ta + (1 - t)b \leq b$. (2 pts.)

5b. For any three real numbers $a \leq c \leq b$, show that there is a $t \in [0, 1]$ such that $c = ta + (1 - t)b$. (4 pts.)

5c. What are the convex subsets of $\mathbb{R}$? (3 pts.)

5c. Show that the emptyset, all the singleton sets and $\mathbb{R}^n$ are convex. (3 pts.)

5d. Is the union of two convex sets convex? Prove or disprove. (3 pts.)

5e. Show that if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are convex, then $X \times Y$ is a convex subset of $\mathbf{R}^{n+m}$. (3 pts.)

5f. Show that the intersection of arbitrarily many convex sets is convex. (4 pts.)

5g. Let X be any subset of $\mathbb{R}^n$. Show that there is a smallest convex subset of $\mathbb{R}^n$ that contains X, i.e. show that there is a subset $C(X) \subseteq \mathbb{R}^n$ such that

i. $X \subseteq C(X)$.

ii. $C(X)$ is convex.

iii. If a subset of $\mathbb{R}^n$ is convex and contains X, then it also contains $C(X)$.

The set $C(X)$ is called the **convex hull** of X. (10 pts.)

5h. What is $C(X)$ if X is convex? (2 pts.)

5i. Assume $X = \{0, 1\} \subseteq \mathbb{R}$. Find $C(X)$. (2 pts.)

5j. Assume $X = \{1/n : n \in \mathbb{N} \setminus \{0\}\}$. Find $C(X)$. (2 pts.)

5k. Assume $X \subseteq \mathbb{R}^2$ contains just two points. What is $C(X)$ geometrically? (2 pts.)

5l. Assume $X \subseteq \mathbb{R}^2$ contains just three points. What is $C(X)$ geometrically? (2 pts.)

5m. Show that if $X \subseteq Y \subseteq \mathbb{R}^n$, then $C(X) \subseteq C(Y)$. (4 pts.)

5n. Show that if $X \subseteq Y \subseteq C(X)$ then $C(Y) = C(X)$. (5 pts.)

5o. Show that $C(X \cap Y) \subseteq C(X) \cap C(Y)$ for all subsets X, Y of $\mathbb{R}^n$. (8 pts.)

5p. Does the reverse inclusion of 3o always hold? (4 pts.)

5q. A subset X of $\mathbb{R}^n$ is called **coconvex** if its complement X^c in $\mathbb{R}^n$ is convex. Show that the union of arbitrarily many coconvex sets is coconvex. (5 pts.)

5r. Show that any subset of $\mathbb{R}^n$ has a largest coconvex subset. (6 pts.)