I. The purpose of this question is to show that the addition (+) is a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{N} \), i.e. to show that the collection \( A = \{(x, y, z) \in \mathbb{N}^3 : z = x + y \} \) is in fact a set.

Recall that + is defined inductively as follows: For all \( x, y \in \mathbb{N} \),
\[
    x + 0 = x \\
    x + S(y) = S(x + y)
\]

Call a subset \( X \) of \( \mathbb{N}^3 \) additive if
i. For all \( x \in \mathbb{N} \), \( (x, 0, x) \in X \)
ii. If \( (x, y, z) \in X \), then \( (x, S(y), S(z)) \in X \).

1. Give an example of an additive set.
2. Show that every member of \( A \) is in every additive set.
3. Show that the intersection of a set of additive sets is an additive set.
4. Show that there is a unique smallest additive set (i.e. an additive set which is a subset of every additive set).
   Let \( B \) be this smallest additive set.
5. Show that every member of \( A \) is in \( B \).
6. Conversely show that if \( (x, y, z) \in B \) then \( z = x + y \). \textbf{Hint:} Proceed by induction on \( y \). Assume \( z \neq x + y \). Show that \( B \setminus \{(x, y, z)\} \) is an additive set.

II. Inspired from the above question, show that the multiplication is a function from \( \mathbb{N} \times \mathbb{N} \) into \( \mathbb{N} \), i.e. to show that the collection \( A = \{(x, y, z) \in \mathbb{N}^3 : z = xy \} \) is in fact a set.