Math 111

Midterm 2
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I. The purpose of this question is to show that the addition (+) is a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, i.e. to show that the collection $A = \{(x, y, z) \in \mathbf{N}^3 : z = x + y\}$ is in fact a set.

Recall that + is defined inductively as follows: For all $x, y \in \mathbb{N}$,

$x + 0 = x$
$x + S(y) = S(x + y)$

Call a subset X of $\mathbb{N}^3$ **additive** if

i. For all $x \in \mathbb{N}$, $(x, 0, x) \in X$
ii. If $(x, y, z) \in X$, then $(x, S(y), S(z)) \in X$.

1. Give an example of an additive set.
2. Show that every member of A is in every additive set.
3. Show that the intersection of a set of additive sets is an additive set.
4. Show that there is a unique smallest additive set (i.e. an additive set which is a subset of every additive set).
Let B be this smallest additive set.
5. Show that every member of A is in B.
6. Conversely show that if $(x, y, z) \in B$ then $z = x + y$. **Hint:** Proceed by induction on y. Assume $z \neq x + y$. Show that $B \setminus \{(x, y, z)\}$ is an additive set.

II. Inspired from the above question, show that the multiplication is a function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$, i.e. to show that the collection $A = \{(x, y, z) \in \mathbb{N}^3 : z = xy\}$ is in fact a set.