

# Valuations

Homework  
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Ali Nesin

1. Let  $\Gamma$  be a multiplicative commutative group. An **ordering** on  $\Gamma$  is a multiplicative subset  $S$  of  $\Gamma$  such that  $\Gamma$  is the disjoint union of  $S$ ,  $S^{-1}$  and  $\{1\}$ . For  $\alpha, \beta \in \Gamma$ , define  $\alpha < \beta$  iff  $\alpha\beta^{-1} \in S$ . Show that  $<$  defines a total order on  $\Gamma$  which is compatible with the group multiplication.

2. Conversely, assume that a total order which is compatible with the group operation is given on a multiplicative commutative group  $\Gamma$ . Show that a multiplicative subset  $S$  of  $\Gamma$  as above gives the same order.

3. Show that a group on which an ordering is given is torsion-free.

If  $\Gamma$  is a group with a valuation, one attaches an element  $0 \notin \Gamma$  to  $\Gamma$  and extends the multiplication and the order of  $\Gamma$  to  $\Gamma \cup \{0\}$  as follows:  $0\alpha = \alpha 0 = 0$  and  $0 < \alpha$  for all  $\alpha \in \Gamma$ .

Let  $K$  be a field. A **valuation** on  $K$  is a map  $|\cdot|$  from  $K$  into  $\Gamma \cup \{0\}$  where  $\Gamma$  is a group with an ordering such that

- i)  $|x| = 0$  iff  $x = 0$ .
- ii)  $|xy| = |x||y|$  for all  $x, y \in K$ .
- ii)  $|x + y| \leq \max(|x|, |y|)$ .

Replacing  $\Gamma$  with  $|K^*|$ , we may (and will) assume that the map  $|\cdot|$  is onto.

4. Show that in a field with valuation  $|1| = 1$  and  $|-x| = |x|$ .

5. Show that in a field with valuation, if  $|x| < |y|$  then  $|x + y| = |y|$ .

6. Let  $(K, |\cdot|, \Gamma)$  be a field with valuation.

6a. Show that  $\mathfrak{o} = \{x \in K : |x| \leq 1\}$  is a local ring with  $\mathfrak{p} = \{x \in K : |x| < 1\}$  as its unique maximal ideal.

6b. Show that for all  $x \in K^*$ , either  $x$  or  $x^{-1}$  is in  $\mathfrak{o}$ .

6c. Show that  $\mathfrak{o}^* = \{x \in K : |x| = 1\} = \mathfrak{o} \setminus \mathfrak{p}$ .

6d. Show that  $\Gamma \approx K^*/\mathfrak{o}^*$  canonically as groups.

6e. By the isomorphism above  $K^*/\mathfrak{o}^*$  can be turned into a group with valuation. What is  $\{s \in K^*/\mathfrak{o}^* : s < 1\}$ ? (This is the subgroup that corresponds to  $S$ ).

7. Let  $K$  be a field. A subring  $\mathfrak{o}$  of  $K$  is called a **valuation ring** if for any  $x \in K^*$ , either  $x$  or  $x^{-1}$  is in  $\mathfrak{o}$ . Let  $\mathfrak{o}$  be a valuation ring of  $K$ .

7a. Show that nonunits of  $\mathfrak{o}$  form an additive subgroup. (**Hint:** Let  $x, y$  be two nonunits of  $\mathfrak{o}$ . We may assume that  $x/y$  is in  $\mathfrak{o}$  (why?). Consider the element  $1 + x/y$  of  $\mathfrak{o}$ ).

7b. Show that the nonunits of  $\mathfrak{o}$  form an ideal  $\mathfrak{p}$  of  $\mathfrak{o}$ .

7c. Show that  $\mathfrak{o}$  is a local ring.

7d. Show that the image of  $\mathfrak{p} \setminus \{0\}$  in  $K^*/\mathfrak{o}^*$  is an ordering in  $K^*/\mathfrak{o}^*$ .

**7e.** For  $x \in K^*$ , let  $|x|$  to be the canonical image of  $x$  in  $K^*/\mathfrak{o}^*$ . Define  $|0| = 0$  ( $0$  is a new element not in  $K^*/\mathfrak{o}^*$ ). Show that  $(K^*, |\cdot|, K^*/\mathfrak{o}^*)$  defines a valuation on  $K^*$ .

**8.** Two valuations  $|\cdot|_1$  and  $|\cdot|_2$  on a field  $K$  are called **equivalent** if there is an order-preserving isomorphism  $\lambda : |K^*|_1 \rightarrow |K^*|_2$  such that  $|x|_2 = \lambda(|x|_1)$  for all  $x \in K$  (we assume that  $\lambda(0) = 0$ ).

Show that there is a one-to-one correspondance between the valuation rings of  $K$  and equivalence classes of valuations.