1. Let $\Gamma$ be a multiplicative commutative group. An **ordering** on $\Gamma$ is a multiplicative subset $S$ of $\Gamma$ such that $\Gamma$ is the disjoint union of $S$, $S^{-1}$ and $\{1\}$. For $\alpha, \beta \in \Gamma$, define $\alpha < \beta$ iff $\alpha \beta^{-1} \in S$. Show that < defines a total order on $\Gamma$ which is compatible with the group multiplication.

2. Conversely, assume that a total order which is compatible with the group operation is given on a multiplicative commutative group $\Gamma$. Show that a multiplicative subset $S$ of $\Gamma$ as above gives the same order.

3. Show that a group on which an ordering is given is torsion-free.

If $\Gamma$ is a group with a valuation, one attaches an element $0 \notin \Gamma$ to $\Gamma$ and extends the multiplication and the order of $\Gamma$ to $\Gamma \cup \{0\}$ as follows: $00 = 0 = \alpha 0$ and $0 < \alpha$ for all $\alpha \in \Gamma$.

Let $K$ be a field. A **valuation** on $K$ is a map $|\cdot|$ from $K$ into $\Gamma \cup \{0\}$ where $\Gamma$ is a group with an ordering such that

i) $|xy| = |x||y|$ for all $x, y \in K$.

ii) $|x + y| \leq \max(|x|, |y|)$.

Replacing $\Gamma$ with $|K^*|$, we may (and will) assume that the map $|\cdot|$ is onto.

4. Show that in a field with valuation $|1| = 1$ and $|-x| = |x|$.

5. Show that in a field with valuation, if $|x| < |y|$ then $|x + y| = |y|$.

6. Let $(K, |\cdot|, \Gamma)$ be a field with valuation.

6a. Show that $\mathfrak{o} = \{x \in K : |x| \leq 1\}$ is a local ring with $\mathfrak{p} = \{x \in K : |x| < 1\}$ as its unique maximal ideal.

6b. Show that for all $x \in K^*$, either $x$ or $x^{-1}$ is in $\mathfrak{o}$.

6c. Show that $\mathfrak{o}^* = \{x \in K : |x| = 1\} = \mathfrak{o} \setminus \mathfrak{p}$.

6d. Show that $\Gamma \cong K^*/\mathfrak{o}^*$ canonically as groups.

6e. By the isomorphism above $K^*/\mathfrak{o}^*$ can be turned into a group with valuation. What is $\{s \in K^*/\mathfrak{o}^* : s < 1\}$? (This is the subgroup that corresponds to $S$).

7. Let $K$ be a field. A subring $\mathfrak{o}$ of $K$ is called a **valuation ring** if for any $x \in K^*$, either $x$ or $x^{-1}$ is in $\mathfrak{o}$. Let $\mathfrak{o}$ be a valuation ring of $K$.

7a. Show that nonunits of $\mathfrak{o}$ form an additive subgroup. (Hint: Let $x, y$ be two nonunits of $\mathfrak{o}$. We may assume that $x/y$ is in $\mathfrak{o}$ (why?). Consider the element $1 + x/y$ of $\mathfrak{o}$).

7b. Show that the nonunits of $\mathfrak{o}$ form an ideal $\mathfrak{p}$ of $\mathfrak{o}$.

7c. Show that $\mathfrak{o}$ is a local ring.

7d. Show that the image of $\mathfrak{p} \setminus \{0\}$ in $K^*/\mathfrak{o}^*$ is an ordering in $K^*/\mathfrak{o}^*$. 

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Valuations
Homework
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Ali Nesin
7e. For \( x \in K^* \), let \( |x| \) to be the canonical image of \( x \) in \( K^*/\mathfrak{o}^* \). Define \( |0| = 0 \) (0 is a new element not in \( K^*/\mathfrak{o}^* \)). Show that \((K^*, | |, K^*/\mathfrak{o}^*)\) defines a valuation on \( K^* \).

8. Two valuations \( |_1 \) and \( |_2 \) on a field \( K \) are called equivalent if there is an order-preserving isomorphism \( \lambda : K^* |_1 \to K^* |_2 \) such that \( |x|_2 = \lambda(|x|_1) \) for all \( x \in K \) (we assume that \( \lambda(0) = 0 \)).

Show that there is a one-to-one correspondence between the valuation rings of \( K \) and equivalence classes of valuations.