Math 111 (Set Theory)

Final Exam (on Ordered Sets)
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Recall that a set together with a binary relation < is called an **ordered set** if
$\forall x\ \neg(x < x)$
$\forall x\ \forall y\ \forall z\ ((x < y \wedge y < z) \to x < z)$
The set of real numbers $\mathbb{R}$ and its subsets (like $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}^{>0}$, $\mathbb{R}^{>0}$) will be considered as ordered sets (ordered with the natural order <).
If S is a set, its set of subsets will be denoted by $\wp(S)$. We will consider $\wp(S)$ as a set ordered by inclusion.

Let $(X, <)$ be an ordered set. A subset A of X is called **dense**[1] in X, if for all $x < y$ in X, there is an $a \in A$ such that $x < a < y$.

1. Is $\mathbb{Z}$ dense in $\mathbb{Q}$? Why?

Answer. No. Because, for example, there is no element of $\mathbb{Z}$ between 1/4 and 3/4.

2. Let S be a nonemty set. Show that $\wp(S)$ has no dense subset.

Answer. If $x \in X$, then there is no subset between $\varnothing$ and $\{x\}$.

Let $A = \{q^2 : q \in \mathbb{Q}\}$.

3. Is A dense in A?

Answer. Yes. If $p^2 < q^2$, then $(p + q)^2/4$ is between them.

4. Is A dense in $\mathbb{Q}^{>0}$? (2 pts.)

Answer. Yes. Let $0 < p < q$ be rational numbers. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is an $r \in \mathbb{Q}$ such that $\sqrt{p} < r < \sqrt{q}$. Now $p < r^2 < q$ and $r^2 \in A$.

5. Is "to be dense in" a transitive relation between ordered sets? I.e., if $A \subseteq B \subseteq C \subseteq X$ and if A is dense in B and B dense in C, is it true that A is dense in C? (3 pts.)

Answer. Yes. Let $c_1 < c_2$ be in C. Then applying the density of B in C twice, we get two elements $b_1, b_2 \in B$ such that $c_1 < b_1 < b_2 < c_2$. Since A is dense in B we get an element a between b_1 and b_2. Now $c_1 < a < c_2$ and this shows that a is dense in C.

6. Is the set $B = \{x/y \in \mathbb{Q} : x, y \in \mathbb{Z}$ are prime to each other and y is odd$\}$ dense in $\mathbb{Q}$?

Answer. Let $p < q \in \mathbb{Q}$. Let n be such that $1/3^n < q - p$. Then there is a maximal natural number a such that $a/3^n \leq p$. Thus $(a+1)/3^n > p$. Also $(a+1)/3^n \leq p + 1/3^n <$

[1] This is not a standard definition.

$p + (q - p) = q$. Thus $p < (a+1)/3^n < q$. Since $(a+1)/3^n \in B$, this shows that B is dense in $\mathbb{Q}$.

7. Is it true that the intersection of two dense subsets of an ordered set is always dense?

Answer. No. Take $\mathbb{R}$ as the ordered set. Then $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$ and their intersection, being empty, cannot be dense in $\mathbb{R}$.

A **morphism** f from an ordered set $(X, <)$ into an ordered set $(Y, \prec)$ is a map $f : X \to Y$ such that for all $a, b \in X$, $a < b$ iff $f(a) \prec f(b)$.

In case $X = Y$ and the orders are the same (i.e. $< = \prec$), a bijective morphism is called an **automorphism**. For example, the identity map is always an automorphism.

8. Show that the inverse map of an automorphism is also an automorphism.

Answer. This is clear from the definition: Let $c, d \in X$. Then $f^{-1}(c) < f^{-1}(d)$ iff $c = f(f^{-1}(c)) < f(f^{-1}(d)) = d$.

9. Show that the composition of two morphisms is also a morphism.

Answer. Clear.

10. Let a and $b \in \mathbb{Q}$. What are the necessary and sufficent conditions on a and b for the map $\varphi_{a,b} : \mathbb{Q} \to \mathbb{Q}$ defined by $\varphi_{a,b}(x) = ax + b$ to be a morphism? Compute $\varphi_{a,b} \circ \varphi_{c,d}$. Find e, f such that $\varphi_{a,b} \circ \varphi_{c,d} = \varphi_{e,f}$.

Answer. Part 1. Note first that $\varphi_{a,b}(x) \in \mathbb{Q}$ for $x \in \mathbb{Q}$, so that $\varphi_{a,b}$ is really a map from $\mathbb{Q}$ into $\mathbb{Q}$. Second, note that $\varphi_{a,b}$ is a morphism iff "$x < y \Leftrightarrow \varphi_{a,b}(x) < \varphi_{a,b}(y) \Leftrightarrow ax < ay$" for all x and y, and this is equivalent to $a > 0$. (This last condition makes $\varphi_{a,b}$ an automorphism).

Part 2. $(\varphi_{a,b} \circ \varphi_{c,d})(x) = \varphi_{a,b}(\varphi_{c,d}(x)) = \varphi_{a,b}(cx + d) = a(cx + d) + b = acx + ad + b = \varphi_{(ac,\, ad + b)}(x)$ for all $x \in \mathbb{Q}$. Thus $\varphi_{a,b} \circ \varphi_{c,d} = \varphi_{(ac,\, ad + b)}$.

11. Define the function $f : \mathbb{Q}^{>0} \to \mathbb{Q}$ as follows:

$$f(q) = \begin{cases} -1/q \text{ if } 0 < q \leq 1 \\ q - 2 \text{ if } 1 \geq q \end{cases}$$

Show that f is an automorphism.

Answer. Note first that f is well-defined since $f(1) = -1$ whether we apply the first or the second part of the definition. The interval $(0, 1]$ is mapped under f bijectively into the interval $(-\infty, -1]$ and the interval $[1, \infty)$ into $[-1, \infty)$. It follows from some simple observations that both pieces are order preserving.

12. Let f be an automorphism of X. Assume X has a least element, say a. Show that $f(a)$ is also a least element of X.

Answer. Assume $y < f(a)$. Let $x \in X$ be such that $f(x) = a$. Then $f(x) = y < f(a)$. Since f is a morphism, we must have $x < a$, theis contradicts the fact that a is a least element of X. Thus there is no such y and $f(a)$ is also a least element of X.

13. Let f be an automorphism of X. Assume X has an element a such that $\{x \in X : a < x\}$ is singleton set. Show that $f(a)$ has the same property.
Answer. Let b the only element which is greater than a. Since $a < b$, we have $f(a) < f(b)$. We will show that $f(b)$ is the only element which is greater than $f(a)$. Assume $f(a) < y$. Let x be such that $f(x) = y$. Then, since $f(a) < f(x)$, we have $a < x$. Thus $x = b$ and so $y = f(x) = f(b)$.

14. Find all automorphism of $\wp(2)$ (here $2 = \{0, 1\}$).
Answer. Let φ be an automorphism of $\wp(2)$. Since $\varnothing$ is the unique smallest element, by number 12, $\varphi(\varnothing) = \varnothing$. For a similar reason (since 2 is the largest element) $\varphi(2) = 2$. Then φ must permute the elements $\{0\}$ and $\{1\}$ of $\wp(2)$. Thus, φ either is identity or interchanges $\{0\}$ and $\{1\}$ and keeps the rest fixed.

15. Find all automorphism of $\wp(3)$.
Answer. There are six of them, one for each permutation of the set $3 = \{0, 1, 2\}$.

16. Let S be a set and let f be a bijection of S. Define $\varphi_f : \wp(S) \to \wp(S)$ by $\varphi_f(A) = f(A)$. Show that φ_f is an automorphism of $\wp(S)$. What is $\varphi_f \circ \varphi_g$? Conversely, show that any automorphism of $\wp(S)$ is of the form φ_f for some bijection f of S.
Answer. Part 1. For $A, B \in \wp(S)$, $A \subset B$ iff $f(A) \subset f(B)$ iff $\varphi_f(A) \subset \varphi_f(A)$. So φ_f is a morphism. Since f is a bijection, φ_f is a bijection as well. Thus φ_f is an automorphism of $\wp(S)$.
Part 2. For all $A \in \wp(S)$, $(\varphi_f \circ \varphi_g)(A) = \varphi_f(\varphi_g(A)) = \varphi_f(g(A)) = f(g(A)) = (f \circ g)(A) = \varphi_{f\circ g}(A)$. Thus $\varphi_f \circ \varphi_g = \varphi_{f\circ g}$.
Part 3. Singleton subsets of S have only one element smaller than them in $\wp(S)$, namely $\varnothing$, and the singleton sets are the only elements of $\wp(S)$ with this property. As in number 13, one can show that an automorphism φ of $\wp(S)$ permutes such subsets of S. Define $f : S \to S$ by the rule $f(x) = y$ iff $\varphi(\{x\}) = \{y\}$. Then $\varphi(\{x\}) = \{f(x)\}$. We claim that $\varphi = \varphi_f$. Let $A \in \wp(S)$. If $x \in A$, then $\{x\} \le A$ and so $\{f(x)\} = \varphi(\{x\}) \le \varphi(A)$. It follows that $f(A) \subseteq \varphi(A)$. Conversely, let $b \in \varphi(A)$. Let a be such that $f(a) = b$. Then $\varphi(\{a\}) = \{f(a)\} = \{b\} \le \varphi(A)$. Thus $\{a\} \le A$, i.e. $a \in A$ and $b = f(a) \in f(A)$.

17. Let $(X, <)$ be a well-ordered set. Show that any morphism $f : X \to X$ satisfies $f(x) \ge x$ for all $x \in X$.
Answer. Assume not. Let x be the least element of X that satisfies $f(x) < x$. Since f is a morphism $f(f(x)) < f(x)$. On the other hand, since $f(x) < x$, by the minimality of x, $f(f(x)) \ge f(x)$, this is a contradiction.

18. Find all automorphisms of $\mathbb{N}$.

Answer. I claim that $\mathrm{Id}_{\mathbb{N}}$ is the only automorphism of $\mathbb{N}$. Let φ be an automorphism of $\mathbb{N}$. We will proceed by induction to show that $\varphi(n) = n$. By number 12, $\varphi(0) = 0$. Assume $\varphi(n) = n$. By number 17, $\varphi(n+1) \geq n + 1$. If $\varphi(n+1) > n + 1$, consider the element x such that $\varphi(x) = n + 1$. We have $\varphi(n+1) > n + 1 = \varphi(x) > n = \varphi(n)$. Thus $\varphi(n+1) > \varphi(x) > \varphi(n)$ and so $n + 1 > x > n$, but there is no such element x in $\mathbb{N}$. Thus $\varphi(n + 1) = n + 1$.

19. Find all automorphisms of $\mathbb{Z}$.

Answer. I claim that the only automorphisms of $\mathbb{Z}$ are given by translations φ_a (for $a \in \mathbb{Z}$) which are defined by $\varphi_a(x) = x + a$. Note first that these are automorphisms since they preserve the order and $\varphi_a^{-1} = \varphi_{-a}$. Next, take any automorphism φ of $\mathbb{Z}$. Set $a = \varphi(0)$. Then $\varphi_{-a} \circ \varphi$ is an automorphism (by number 9) and it sends 0 to 0. We claim that an automorphism ψ of $\mathbb{Z}$ that sends 0 to 0 is the identity map. Since $\psi(0) = 0$, ψ must send $\mathbb{N}$ to $\mathbb{N}$, thus, by number 18, the map ψ restricted to $\mathbb{N}$ must be the identity map. As in number 18, one can show that $\psi(-a) = -a$ for all $a > 0$. Thus ψ is the identity map. In particular $\varphi_{-a} \circ \varphi = \mathrm{Id}_{\mathbf{Z}}$, and so $\varphi = \varphi_{-a}^{-1} = \varphi_a$.

20. Find a morphism from $\mathbb{Z}$ into $\mathbb{Z}$ which is not an automorphism.

Answer. $\varphi(x) = 2x$ is such an example.

A surjective morphism is called **isomorphism**. Two ordered sets are called **isomorphic** if there is an isomorphism between them.

21. Are $\mathbb{Z}$ and $\mathbb{Q}$ isomorphic?

Answer. No, because $\mathbb{Q}$ is dense and not $\mathbb{Z}$.

22. Are $\mathbb{R}^{\geq 0}$ and $\mathbb{R}$ isomorphic?

Answer. No, because $\mathbb{R}^{\geq 0}$ has a least element and not $\mathbb{R}$.

23. Are $\mathbb{R}^{>0}$ and $\mathbb{R}$ isomorphic? (4 pts.)

Answer. Yes, the map $x \to e^x$ is one of the several automorphisms from $\mathbb{R}$ onto $\mathbb{R}^{>0}$.

24. Are the open interval (0, 1) and $\mathbb{R}$ isomorphic? (5 pts.)

Answer. Yes.

The map $x \to e^x$ is an automorphism from $\mathbb{R}$ onto $\mathbb{R}^{>0}$.

The map $x \to -1/(1+x^2)$ is an automorphism from $\mathbb{R}^{>0}$ onto $(-1, 0)$.

The map $x \to x + 1$ is an automorphism from $(-1, 0)$ onto $(0, 1)$.

Compose these, to get an automorphism from $\mathbb{R}$ onto (0, 1). One finds $f(x) = e^{2x}/(1+e^{2x})$

25. Show that $\mathbb{Q}$ has uncountably many automorphisms.

Answer. Note first that for any $k, a, b \in \mathbb{Q}$, if $a < b$, by number 10, the map

$$x \mapsto a - k(b - a) + (b - a)x$$

is an automorphism from the interval $[k, k + 1]$ onto the interval $[a, b]$. Let $\sigma = (a_n)_{n \in \mathbf{N}}$ be any strictly increasing sequence of natural numbers such that $q_0 = 0$. Define $f_\sigma : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$f_\sigma(x) = \begin{cases} x & \text{if } x < 0 \\ a_n + (b_n - a_n)(x - n) & \text{if } n \le x \le n+1 \end{cases}$$

Then f_σ is an automorphism of $\mathbb{Q}$. Thus there are at least as many automorphisms as strictly increasing sequences of natural numbers. But there are as many strictly increasing sequence of natural numbers whose first eleemnt is 0 as there are infinite subsets of $\mathbb{N} \setminus \{0\}$. Since the number of finite subsets of $\mathbb{N} \setminus \{0\}$ is countable (prove it!), there are uncountably many infinite sequences of $\mathbb{N}$ whose first term is 0.

26. Order $\mathbb{N} \times \mathbb{N}$ as follows: $(x, y) \le (z, t)$ iff $x \le z$ and $y \le t$ (then define $(x, y) < (z, t)$ iff $(x, y) \le (z, t)$ and $(x, y) \ne (z, t)$). Show that the map $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $\alpha(x, y) = (y, x)$ is an automorphism of $\mathbb{N} \times \mathbb{N}$. Show that $\mathrm{Id}_\mathbb{N}$ and α are the only automorphisms of $\mathbb{N} \times \mathbb{N}$.

Answer. The map α is easily seen to be an automorphism of $\mathbb{N} \times \mathbb{N}$. Let φ be any automorphism of $\mathbb{N} \times \mathbb{N}$. Since $(0, 0)$ is the only smallest element, $\varphi(0, 0) = (0, 0)$. Since $(1, 0)$ and $(0, 1)$ are the only elements of $\mathbb{N} \times \mathbb{N}$ which are just greater than $(0, 0)$, φ must either fix or interchange them. By composing with α if necessary, we may assume that φ fixes $(0, 0)$, $(1, 0)$ and $(0, 1)$. We will show that such a φ must be the identity. We proceed by induction on $x + y$ to show that $\varphi(x, y) = (x, y)$. By induction $\varphi(x, y - 1) = (x, y - 1)$ and $\varphi(x - 1, y) = (x - 1, y)$. There are only two elements which are just greater than $(x, y - 1)$ and these are $(x + 1, y)$ and (x, y). Similarly there are only two elements which are just greater than $(x - 1, y)$ and these are $(x, y + 1)$ and (x, y). Thus (x, y) is the only element which is just greater than both $(x, y - 1)$ and $(x - 1, y)$. Thus (x, y) must be fixed as well.

27. Order $\mathbb{Z} \times \mathbb{Z}$ as above. Let $a, b \in \mathbb{Z}$. Show that the map $\tau_{a,b}$ defined by

$$\tau_{a,b}(x, y) = (x + a, y + b)$$

defines an automorphism of $\mathbb{Z} \times \mathbf{Z}$. Show that the set $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ of automorphisms of $\mathbb{Z} \times \mathbb{Z}$ is

$$\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}) = \{\tau_{a,b} : a, b \in \mathbb{Z}\} \cup \{\alpha \circ \tau_{a,b} : a, b \in \mathbb{Z}\}$$

where α is defined as above.

Answer. The first part is easy. Let φ be an automorphism of $\mathbb{Z} \times \mathbb{Z}$. Let $\varphi(0, 0) = (a, b)$. Then $\tau_{(-a,\,-b)} \circ \varphi$ is an automorphism that sends $(0, 0)$ to $(0, 0)$. Thus it sends $\mathbb{N} \times \mathbb{N}$ onto itself. It follows that its restriction to $\mathbb{N} \times \mathbb{N}$ is an automorhism of $\mathbb{N} \times \mathbb{N}$. By number 26, its restriction to $\mathbb{N} \times \mathbb{N}$ is either $\mathrm{Id}_{\mathbb{N}\times\mathbb{N}}$ or α. Thus either $\tau_{(-a,\,-b)} \circ \varphi$ or $\alpha \circ \tau_{(-a,\,-b)} \circ \varphi$ is identity on $\mathbb{N} \times \mathbb{N}$. Then as in number 26, one can show that such an automorphism is identity on $\mathbb{Z} \times \mathbb{Z}$. It follows $\varphi = \tau_{(-a,\,-b)}$ or $\varphi = \tau_{(-a,\,-b)} \circ \alpha = \alpha \circ \tau_{(-a,\,-b)}$.