Math 111

Midterm 3
February 2000
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1. Show that $\lim_{n\to\infty} (1/2^0 + 1/2^1 + 1/2^2 + ... + 1/2^n) = 2$. (5 points)

2. Let $x_n = 1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2$.

2a. Show that for all integers $n \geq 1$, $x_n \leq 2 - 1/n$. (5 points)

2b. Conclude that the sequence $(x_n)_n$ is Cauchy. (5 points)

2c. Let $k \geq 2$ be an integer. Show that the sequence $(x_n)_n$ where

$$x_n = 1/1^k + 1/2^k + 1/3^k + ... + 1/n^k$$

is Cauchy. (5 points)

2d. Show that for an integer n large enough, $n^2 \leq 2^n$. (10 points)

2e. Conclude that the sequence $(x_n)_n$ is bounded below by $1 + 1/4 + 1/9 + 1/8$ ($= 107/72$). (5 points)

3. Show that the sequence $y_n = 1 - 1/2 + 1/3 - ... + (-1)^{n-1}/n$ is a Cauchy sequence (Hint: Apply the definition, try to group the terms two by two and use 2b). (20 pts.)

4. The purpose of this exercise is to show that there is a square rational number between any two distinct positive rational numbers. Below we give hints. But you may try to solve it without using the hints, in which case you still get the maximum number of points, which is 30.

Hints: Let $n > 0$ be any natural number. Given a natural number $x > 0$, let $f(x)$ be the largest natural number such that $f(x)^2/x^2 < n$.

4a. Show that there is a natural number x such that $4n + 2 < x^2 + 1/x^2$. (5 points)

4b. Let x be as above. Show that $(f(x) + 1)^2/x^2 - f(x)^2/x^2 \leq 1$. (10 points)

4c. Conclude that $n \leq (f(x) + 1)^2/x^2 \leq n + 1$. (5 points)

4d. Conclude from above that if $0 \leq a < b$ are rational numbers, then there is a rational number q such that $a \leq q^2 \leq b$. (10 points)

5a. Let $(x_n)_n$ be a sequence of positive rational numbers. Assume that $(x_n^2)_n$ is Cauchy. Show that $(x_n)_n$ is Cauchy. (25 pts.)

5b. Show that there is a Cauchy sequence $(q_n)_n$ such that $\lim_{n\to\infty} q_n^2 = 2$. **(Hint:** Use #4). (10 pts.)