Math 111 (Set Theory)

Second Midterm
Ali Nesin – Özlem Beyarslan
January 2000

1. Let $a \in \mathbb{Q}$ be fixed. We will say that two functions f and g from $\mathbb{Q}$ into $\mathbb{Q}$ **have the same germ around** a, and we write $f \equiv_a g$, if there is $\varepsilon \in \mathbb{Q}^{>0}$ such that $f(x) = g(x)$ for all $x \in (a - \varepsilon, a + \varepsilon)$.

1a. Show that $\equiv_a$ is an equivalence relation on the set ${}^{\mathbb{Q}}\mathbb{Q}$ of all functions from $\mathbb{Q}$ into $\mathbb{Q}$. For $f \in {}^{\mathbb{Q}}\mathbb{Q}$, let $[f]$ denote the equivalence class of f with respect to this equivalence relation.

1b. For f and g in ${}^{\mathbb{Q}}\mathbb{Q}$, we define $f + g$ and fg as functions from $\mathbb{Q}$ into $\mathbb{Q}$ as follows: For all $x \in \mathbb{Q}$,

$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x).$$

Show that if $f_1 \equiv_a g_1$ and $f_2 \equiv_a g_2$ then $f_1 + f_2 \equiv_a g_1 + g_2$ and $f_1f_2 \equiv_a g_1g_2$. It follows that one can define addition and multiplication on ${}^{\mathbb{Q}}\mathbb{Q}/\equiv_a$.

1c. For $r \in \mathbb{Q}$, define $c_r : \mathbb{Q} \to \mathbb{Q}$ by $c_r(x) = r$ for all $x \in \mathbb{Q}$, i.e. c_r is the constant function that takes always the value r. Show that the function $c : \mathbb{Q} \to {}^{\mathbb{Q}}\mathbb{Q}/\equiv_a$ defined by $c(r) = [c_r]$ is one-to-one and that it satisfies the equalities

$$c(r + s) = c(r) + c(s)$$
$$c(rs) = c(r)c(s)$$

for all $r, s \in \mathbb{Q}$.

1d. For $f \in {}^{\mathbb{Q}}\mathbb{Q}$ show that the following two conditions are equivalent:

(*i*) For some $\varepsilon \in \mathbb{Q}^{>0}$, $f(x) \neq 0$ for all $x \in (a - \varepsilon, a + \varepsilon)$.

(*ii*) There is a $g \in {}^{\mathbb{Q}}\mathbb{Q}$ such that $[f][g] = [c_1]$.

1e. Inspired by 1a define the equivalence relation $\equiv_\infty$.

2a. Let $(a_n)_n$ and $(b_n)_n$ be two rational sequences that converge to two different numbers. Show that $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\}$ is finite.

2b. Let $(a_n)_n$ and $(b_n)_n$ be two rational Cauchy sequences such that the set $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\}$ is infinite. Show that $\lim_{n \to \infty} (a_n - b_n) = 0$.

2c. Let $(a_n)_n$ and $(b_n)_n$ be two Cauchy sequences such that the set $\{n \in \mathbb{N} :$ there is an m such that $a_n = b_m\}$ is infinite. Prove or disprove: $\lim_{n \to \infty} (a_n - b_n) = 0$.

3. Let A and B be two sets of size n and m respectively. How many one-to-one functions are there from A into B?

4a. Show that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for all n and k such that $k < n$.

4b. Show that $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ for all $n \in \mathbb{N}$ and x and $y \in \mathbb{Q}$. (Hint: By induction on n).

4c. Let p be a prime number. Show that p divides $\binom{p}{i}$ for all $i = 1, \ldots, p-1$.

4d. Show that for all primes p and natural numbers n, p divides $n^p - n$. (Hint: By induction on n).