Set Theory

First Midterm
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I. Recall the Peano Axioms:

PA1. The function $S : \omega \to \omega$ given by $S(x) = x^+ = x \cup \{x\}$ is one-to-one.

PA2. $S(\omega) = \omega \setminus \{0\}$.

PA3. If $X \subseteq \omega$ is such that

a) $0 \in X$

b) if $x \in X$ then $S(x) \in X$

then $X = \omega$.

Recall also the definition of + and ×: For all $x, y \in \omega$,

S1. $x + 0 = x$

S2. $x + S(y) = S(x + y)$

P1. $x0 = 0$

P2. $xy^+ = xy + x$

I.1. Show that $x + y = y + x$ for all $x, y \in \omega$.

I.2. Show that $xy = yx$.

II. For this question, you may assume that you know the basic rules of arithmetic for the structure $(\omega, +, \times)$.

On the set $\omega \times \omega$ we define the relation $\equiv$ as follows:

$$(a, b) \equiv (c, d) \text{ iff } a + d = b + c$$

II.1. Show that this is an equivalence relation on $\omega \times \omega$.

II.2. Find the equivalence classes of (0, 0), (1, 0), (0, 1), (2, 6), (1, 5) and (a, b).

II.3. Show that if $(a, b) \equiv (c, d)$ and $(a', b') \equiv (c', d')$ then $(a + a', b + b') \equiv (c + c', d + d')$ and $(aa' + bb', ab' + ba') \equiv (cc' + dd', cd' + dc')$.

II.4. Let $[a, b]$ be the class of (a, b) and let $\mathbb{Z} = \omega \times \omega/\equiv$ be the set of equivalence relations. Say why question #3 allows us to define + and × on the set $\mathbb{Z}$ as:

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] \times [c, d] = [a, b][c, d] = [ac + bd, ad + bc]$$

II.5. Show that there is a one-to-one map f from ω into $\mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$,

$$f(x + y) = f(x) + f(y)$$
$$f(xy) = f(x)f(y).$$

II.6. What do you think about $\mathbb{Z}$?

Grading Policy:
I.1. 30 pts.
I.2. 30 pts.
II.1. 8 pts.
II.2. 6 pts.
II.3. 12 pts.
II.4. 5 pts.
II.5. 15 pts.
II.6. 4 pts.