Set Theory Final Exam

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Write clearly and without using mathematical symbols like ∀, ∃, ⇔ etc.

1. Prove that for $n \in \mathbb{N}$, the set $\wp(n)$ of subsets of n has 2^n elements. (**Hint:** You may proceed by induction on n).

2a. Show that a subset of $\mathbb{R}$ which is closed under substraction is closed under addition.

2b. Does the converse of the above statement hold?

2c. Show that a subset of $\mathbb{R}$ which is closed under substraction, squaring and dividing by 2 is closed under multiplication.

3. Let us call a subset X of $\mathbb{R}$ **square-closed** if for all $x \in X$, $x^2 \in X$.

3a. Show that any subset A of $\mathbb{R}$ is contained in a smallest square-closed set.

3b. Show that $\mathbb{R}$ has a maximal square-closed subset that does not contain 1.

4. Let X be any set. Show that there is no one-to-one correspondance between X and the power set $\wp(X)$ of X. (**Hint:** Assuming there is such a bijection f from X onto $\wp(X)$, consider the set $A = \{x \in X : x \in f(x)\}$).

5. Define the set ℓ^∞ to be the set of all bounded sequences of $\mathbb{R}$. Define the addition, the substraction and the multiplication of sequences componentwise.

5a. Show that ℓ^∞ is closed under addition, substraction and multiplication.

5b. Show that there is a one-to-one map from $\mathbb{R}$ into ℓ^∞ that respects addition, substraction and multiplication, i.e. that there is a one-to-one map i from $\mathbb{R}$ into ℓ^∞ such that for all $x, y \in \ell^\infty$, $i(x + y) = i(x) + i(y)$, $i(x - y) = i(x) - i(y)$ and $i(xy) = i(x)i(y)$.

For $x = (x_n)_n$ and $y = (y_n)_n$ in ℓ^∞, define $|x| = \sup_{n\in\mathbb{N}} |x_n|$ and the **distance** $d(x, y)$ from x to y as $d(x, y) = |x - y|$.

5c. Show that for $x, y \in \ell^\infty$, $d(x, y) = 0$ iff $x = y$.

5d. Show that for $x, y \in \ell^\infty$, $d(x, y) = d(y, x)$.

5e. Show that for $x, y, z \in \ell^\infty$, $d(x, z) \leq d(x, y) + d(y, z)$.

A sequence $(x_n)_n$ of elements of ℓ^∞ (i.e. each $x_n = (x_{np})_p$ is a bounded sequence of real numbers) is called a Cauchy sequence if for all real numbers $\varepsilon > 0$ there is a natural number N such that for all $n, m > N$, $|x_n - x_m| < \varepsilon$.

5f. Show that if a sequence $(x_n)_n$ of elements of ℓ^∞ is Cauchy then for all $p \in \mathbb{N}$, the sequence $(x_{np})_n$ of real numbers is Cauchy.

5g. Does the converse of the above statement hold?