Set Theory

Math 111
Resit
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Always justify your answer. A short answer of the form "yes" or "no" will not be accepted.

1. A subset A of an ordered set $(X, <)$ is called **dense** in X, if for all $x < y$ in X, there is an $a \in A$ such that $x < a < y$. Let $A = \{q^2 : q \in \mathbb{Q}\}$. Order A naturally (the induced order of $\mathbb{Q}$). Is A dense in $\mathbb{Q}$? (5 pts.)

2. Let X and Y be two sets and $f : X \to Y$ be a function. Let $A_i \subseteq X$ and $B_j \subseteq Y$.

2a. Show that $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$. (4 pts.)

2b. Show that $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$. (2 pts.)

2c. Show that $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$. (4 pts.)

2d. Show that $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$. (4 pts.)

2e. Does the reverse inclusion in #2b hold? Prove or give a counterexample. (4 pts.)

3. Find a map f: $\mathbb{R} \to \mathbb{R}$ such that $\bigcap_{i=1}^{n} f^i(\mathrm{R}) \neq \varnothing$ but $\bigcap_{i\in \mathbf{N}} f^i(\mathbf{R}) = \varnothing$. (8 pts.)

4. Let X be a nonempty subset of $\mathbb{R}$ such that for $x, y \in X$, $x - y \in X$.

4a. Show that $0 \in X$. (1 pt.)

4b. Show that for $x \in X$, $-x \in X$ also. (1 pts.)

4c. Show that $x + y \in X$ for $x, y \in X$. (2 pts.)

From now on assume further that for $x \in X$, $x^2 \in X$ also.

4d. Show that for $x, y \in X$, $2xy \in X$ also. (3 pts.)

4e. Show that $2X$ is closed under $+$, $-$ and $\times$. (4 pts.)

4f. Show that the relation

$$x \equiv y \text{ iff } x - y \in 2X$$

defines an equivalence relation on X. (4 pts.)

5. We will call a subset X of $\mathbb{R}$ square-closed if for all $x \in X$, $x^2 \in X$ also. Note that $\varnothing$ and $\mathbb{R}$ are square closed subsets of $\mathbb{R}$.

5a. Show that if Π is a set of square-closed subsets of $\mathbb{R}$, then $\cup\Pi$ and $\cap\Pi$ are square closed subsets of $\mathbb{R}$. (4 pts.)

5b. Let A be any subset of $\mathbb{R}$. Show that there is a smallest square-closed subset A^* that contains A. (4 pts.)

5c. Let A be any subset of $\mathbb{R}$. Show that there is a largest square-closed subset A° of A. (4 pts.)

5d. Prove or disprove: For any two subsets A and B of $\mathbb{R}$,

$$A^* \cup B^* = (A \cup B)^*$$
$$A^* \cap B^* = (A \cap B)^*$$
$$A^\circ \cup B^\circ = (A \cup B)^\circ$$
$$A^\circ \cap B^\circ = (A \cap B)^\circ$$

(12 pts.)

6. [Cantor-Schröder-Bernstein] Let A be a set and A' a subset of A. Assume that there is a bijection $f: A \to A'$ betweeen A and A'. Let B be any set such that $A' \subseteq B \subseteq A$. The purpose of this exercise is to show that there is a bijection between B and A.

Let $Q = B \setminus A'$.

Let $\Gamma = \{X \subseteq A : Q \cup f(X) \subseteq X\}$.

Let $T = \cap\, \Gamma = \bigcap_{X \in \Gamma} X$.

6a. Show that $T \in \Gamma$. (4 pts.)

6b. Show that $Q \cup f(T) \in \Gamma$. (4 pts.)

6c. Show that $T = Q \cup f(T)$. (Hint: Use a and b). (5 pts.)

6d. Show that $B = T \cup (A' \setminus f(T))$. (Hint: Use c). (5 pts.)

6e. Show that $T \cap (A' \setminus f(T)) = \varnothing$. (5 pts.)

6f. Show that there is a bijection between B and A. (Hint: Use parts d and e). (5 pts.)