Set Theory

Summer Final I
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Always justify your answer. A short answer of the form "yes" or "no" will not be accepted.

The total number of points is more than 100, but a 100 will considered as a perfect score.

1. Let $(X, <)$ be an ordered set. A subset A of X is called **dense** in X, if for all $x < y$ in X, there is an $a \in A$ such that $x < a < y$. In this exercise, we will mainly consider the ordered set $\mathbb{Q}$ (ordered naturally).

1a. Is $\mathbb{Z}$ dense in $\mathbb{Q}$? (2 pts.)

1b. Let S be any set. Consider the set $\wp(S)$ of subsets of S as an ordered set by the inclusion relation. Show that $\wp(S)$ has no dense subset. (3 pts.)

1b. Let $A = \{q^2 : q \in \mathbb{Q}\}$. Order A naturally (the induced order of $\mathbb{Q}$). Is A dense in A? (2 pts.)

1c. Is A dense in $\mathbb{Q}^{>0}$? (2 pts.)

1c. Is "to be dense in" a transitive relation between ordered sets? I.e., if $A \subseteq B \subseteq C \subseteq X$ and if A is dense in B and B dense in C, is it true that A is dense in C? (6 pts.)

1d. Let $A = \{x/y \in \mathbb{Q} : x, y \in \mathbb{Z}$ are prime to each other and y is odd$\}$. Is A dense in $\mathbb{Q}$? (6 pts.)

2. Define the function $f : \mathbb{Q}^{>0} \to \mathbb{Q}$ as follows:

$$f(q) = \begin{cases} -1/q \text{ if } 0 < q < 1 \\ q - 2 \text{ if } 1 > q \end{cases}$$

Show that f is an order preserving bijection. (4 pts.)

3a. Are the ordered sets $\mathbb{Z}$ and $\mathbb{Q}$ isomorphic? (2 pts.)

3b. Are the ordered sets $\mathbb{Q}$ and $\mathbb{R}$ isomorphic? (2 pts.)

3c. Are the ordered sets $\mathbb{R}^{\geq 0}$ and $\mathbb{R}$ isomorphic? (3 pts.)

3d. Are the ordered sets $\mathbb{R}^{>0}$ and $\mathbb{R}$ isomorphic? (4 pts.)

3e. Are the ordered sets $(0, 1)$ and $\mathbb{R}$ isomorphic? (5 pts.)

3f. Which of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ is well-ordered? (3 pts.)

4. Let $(X, <)$ be a well-ordered set. Show that any order preserving $f : X \to X$ satisfies $f(x) \geq x$ for all $x \in X$. (7 pts.)

5. What is the cardinality of the set of cofinite subsets of $\mathbb{N}$. (3 pts.)

6. Is it true that any set of cardinal numbers has a maximal element? (8 pts.)

7a. Let $f : \mathbb{Q} \to \mathbb{Q}$ respect the order and the addition, i.e.,

$$\forall x, y\ (x < y \to f(x) < f(y) \text{ and } f(x + y) = f(x) + f(y)).$$

Show that there is an $a > 0$ such that $f(x) = ax$. (10 pts.)

7b. Assume further that f respects the multiplication. Show that $f = \mathrm{Id}_{\mathbb{Q}}$. (3 pts.)

7c. Let $b \in \mathbb{Q}$ and $a \in \mathbb{Q}^{>0}$. Show that $f(x) = ax + b$ is an order preserving bijection. (2 pts.)

7d. Find an order preserving bijection $f : \mathbb{Q} \to \mathbb{Q}$ which is different from the ones in 7c. (4 pts.)

8. Recall that ε_o is the least ordinal larger than all $\omega^{\omega^{\cdot^{\cdot^{\cdot\omega}}}}$ (n times) for all n. What is the cardinality of ε_o? (8 pts.)

9a. Let $\mathfrak{R} = \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R} \cup \{\infty, -\infty\} \text{ and } a < b\}$. (Here we assume that $-\infty < x < \infty$ for all $x \in \mathbb{R}$). Without using the axiom of choice, find a function $f: \mathfrak{R} \to \mathbb{R}$ such that $f(I) \in I$ for all intervals $I \in \mathfrak{R}$. (10 pts.)

9b. Without using the axiom of choice, find a function $f: \mathfrak{R} \to \mathbb{Q}$ such that $f(I) \in I$ for all intervals $I \in \mathfrak{R}$. (10 pts.)

10. Let $\mathfrak{R}$ be a set of nonempty disjoint sets. Show that $\mathrm{Card}(\cup\mathfrak{R}) \geq \mathrm{Card}(\mathfrak{R})$. (10 pts.)

11. A function $f : \mathbb{N} \to \mathbb{N}$ is called **decidable** if one can write a (finite of course) computer program CP(f) that calculates $f(x)$ with the input x for all $x \in \mathbb{N}$ after a finite amount of time. Otherwise, a function is called **undecidable**. Show that there is an undecidable function. (15 pts.)

12. Let X be a set. A **filter** on X is a set $\mathfrak{I}$ of subsets of X that satisfies the following properties:

i) If $A \in \mathfrak{I}$ and $A \subseteq B \subseteq X$, then $B \in \mathfrak{I}$.

ii) If A and B are in $\mathfrak{I}$, then so is $A \cap B$.

iii) $\emptyset \notin \mathfrak{I}$ and $X \in \mathfrak{I}$.

If $A \subseteq X$ is a fixed nonempty subset of X, then the set of subsets of X that contain A is a filter on X, called **principal filter**.

If X is infinite, then the set of cofinite subsets of X is a filter, called **Fréchet filter**.

12a. Show that the Fréchet filter is not principal. (5 pts.)

A filter is called **ultrafilter** if it is a maximal filter.

12b. Show that if X is infinite then there are nonprincipal ultrafilters on X. (15 pts.)

13a. Show that any two dense and countable total orders without minimal and maximal elements are isomorphic. (**Hint:** Write the sets as $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$. Construct the isomorphism with a "back and forth argument".) (20 pts.)

13b. Conclude that the ordered sets $\mathbb{Q}$ and $\mathbb{Q} \cup \{\sqrt{2}\}$ are isomorphic. (3 pts.)

13c. Show that the ordered sets $\mathbb{Q} \cup (0, 1)$ and $\mathbb{Q} \cup (0,1) \cup (2, 3)$ are not isomorphic. (Here the intervals are real intervals). (7 pts.)