MATH 111

Summer Midterm I
14th of June, 1999
Ali Nesin

1. Show that Zorn's Lemma implies Axiom of Choice in view of the other axioms of set theory. (20 pts.)

A totally ordered set $(X, \leq)$ is called a **well-ordered set** if every nonempty subset of X has a least element, i.e. if for all $\emptyset \neq A \subseteq X$, there is an $a \in A$ such that $a \leq x$ for all $x \in A$. A natural number and the set ω of natural numbers are examples of well-ordered sets.

2. On ω define

$$a \prec b \text{ iff } b \geq a.$$

Show that $(\omega, \prec)$ is not a well-ordered set. (5 pts.)

3. On $\omega \times \omega$ define

$$(a, b) \leq (x, y) \text{ iff } (2a + 1)2^y \leq (2x + 1)2^b.$$

Show that this defines a total order on $\omega \times \omega$ which is not a well-order. (15 pts.)

4. On $\omega \times \omega$ define

$$(a, b) \leq (x, y) \text{ iff either } a < x \text{ or } (a = x \text{ and } b \leq y)..$$

Show that this defines a well-ordered set. (10 pts.)

5. Show that there is no strictly decreasing function from ω into into a well-ordered set. (10 pts.)

6. Well-ordering Theorem. Show that every set can be well-ordered. (Hint: Use the Axiom of Choice. (20 pts.)

7. Show that the Well-Ordering Theorem implies the Axiom of Choice in view of the other axioms of set theory. (Hint: Show that choice functions exist). (20 pts.)