Group Theory Problems
(Oleg Belegradek)

1. Prove that
   (a) $(\mathbb{C}(n), \cdot) \cong (\mathbb{Z}_n, +)$,
   (b) $(\mathbb{R}, +) \cong (\mathbb{R}^{>0}, \cdot)$,
   (c) $(\mathbb{Q}, +) \ncong (\mathbb{Q}^{>0}, \cdot)$,
   (d) $GL(F^n) \cong GL_n(F)$ (F is a field).

2. Show that
   (a) $S_n = \langle (12), (13), \dots, (1n) \rangle = \langle (12), (12 \dots n) \rangle$,
   (b) $A_n = \langle (ijk) : i, j, k \text{ are distinct} \rangle$.

3. Let K be a field, $n \geq 2$. Let $t_{ij}(\alpha) = e + \alpha e_{ij}$, with $i \neq j$, $d(\beta) = e + (\beta - 1)e_{nn}$. Prove that any matrix can be written in the form $t_1 t_2 \dots t_n d(\beta) t_{n+1} \dots t_s$ where each $t_k = t_{ij}(\alpha)$ for some $\alpha \in K$. Show that
   (a) $GL_n(K) = \langle t_{ij}(\alpha), d(\beta) : \alpha, \beta \in K,\ \beta \neq 0,\ i \neq j \rangle$,
   (b) $SL_n(K) = \langle t_{ij}(\alpha) : \alpha \in K,\ i \neq j \rangle$,
   (c) $T_n(K) = \langle t_{ij}(\alpha), diag(\beta_1, \dots, \beta_n) : i < j,\ \beta_1, \dots, \beta_n \neq 0 \rangle$,
   (d) $UT_n(K) = \langle t_{ij}(\alpha) : i < j \rangle$.

4. Let G be a group, $g \in G$. Show that
   (a) If $g^n = 1$ then $|g|$ divides n.
   (b) $|g| = |\langle g \rangle|$.

5. Let G be a group, $a, b \in G$. Assume $ab = ba$ and $|a|, |b| < \infty$. If $(|a|, |b|) = 1$, then $|ab| = |a|\,|b|$. If $(|a|, |b|) \neq 1$, then there exists a $c \in G$ such that $|c| = \operatorname{lcm}(|a|, |b|)$.

6. Show that for any $n, m, k \in \{2, 3, 4, \dots\} \cup \{\infty\}$, there is a group G and $a, b \in G$ such that $|a| = n$, $|b| = m$ and $|ab| = k$.

7. Show that all subgroups of $\mathbb{C}(p^\infty)$ are $\mathbb{C}(p^n)$.

8. Prove that any abelian group without proper infinite subgroups is isomorphic to $\mathbb{C}(p^\infty)$.

9. Find the following groups explicitly:
   (a) $Aut(\mathbb{Z}, +)$
   (b) $Aut(\mathbb{Z}, <)$
   (c) $Aut(\mathbb{Z}, +, <)$

(d) $Aut(\{1,\ldots,5\},\{(1,3),(2,3),(4,3),(5,3)\})$ (as a graph)
(e) $Isom(\mathbb{R})$
(f) $Isom(\mathbb{R}^{\geq 0})$
(g) $Isom(\mathbb{R}^{\geq 0} \cup i\mathbb{R}^{\geq 0})$

10. Let $G = \overline{\prod}_{i\in I} G_i$. Let $J \subseteq I$, $G_J = \{f \in G : \forall i \in J^c \;\; f(i) = e_i\}$. Show that $G \cong G_J \times G_{J^c}$. Prove the same statement for direct sum.

11. Show that $D_n(K) \cong \overline{\prod}_n K^*$.

12. Prove that if $\gcd(n,m) = 1$, then $\mathbb{C}(n) \times \mathbb{C}(m) \cong \mathbb{C}(nm)$.

13. Find all left and right cosets of $\langle(12)\rangle$ in $Sym(3)$.

14. Show that $|\mathbb{Z} : \langle n\rangle| = n$.

15. Prove that $\mathbb{Q}$ has no proper subgroup of finite index.

16. Show that $|Sym(n) : Alt(n)| = 2$.

17. Let $A, B \leq G$. Prove that $|G : A \cap B| \leq |G : A|$.

18. Let $A \leq B \leq G$. Show that $|G : A| < \infty$ if and only if $|G : B|$, $|G : A| < \infty$. If we have this situation then $|G : A| = |G : B| \,.\, |B : A|$.

19. Let $A, B \leq G$ be of finite index. Show that $|G : A \cap B| \leq |G : A| \,.\, |G : B|$.

20. Prove that

    (a) $Inn(G) \trianglelefteq Aut(G)$
    (b) $SL_n(K) \trianglelefteq GL_n(K)$, K is a commutative ring with unit.
    (c) $UT_n(K) \trianglelefteq T_n(K)$
    (d) $UT_n^m(K) \trianglelefteq T_n(K)$

21. Show that every subgroup of index 2 is normal.

22. Let $A, B \leq G$. Show that

    (a) if $A \trianglelefteq G$ then $AB = BA \leq G$.
    (b) if $A, B \trianglelefteq G$ then $AB \trianglelefteq G$.
    (c) it is possible that $A \trianglelefteq B$ and $B \trianglelefteq G$ but $A \ntrianglelefteq G$.

23. G is abelian if an only if for all $g \in G \;\; \left|g^G\right| = 1$.

24. Suppose that $A \leq B \leq G$. Show that

    (a) $N_B(A) \leq B$
    (b) $A \trianglelefteq B$ if and only if $N_B(A) = B$

25. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Q}, a \neq 0 \right\}.$$

Find a subgroup of G which is conjugate to a proper subgroup of it.

26. Show that $Z(G) \trianglelefteq G$.

27. Show that

    (a) $Sym(n)$ is abelian for $n < 3$.

    (b) $Z(Sym(n)) = \{Id\}$ if $n \geq 3$.

    (c) $Alt(1) = Alt(2) = \{Id\}$, $Alt(3) = \langle (123) \rangle$.

    (d) $Z(Alt(n)) = \{Id\}$ for $n \geq 4$.

28. Let K be a field. Prove the following

    (a) $Z(GL_n(K)) = \{\alpha E : \alpha \in K^*\}$

    (b) $Z(SL_n(K)) = \{\alpha E : \alpha^n = Id\}$

    (c) $Z(T_n(K)) = \{\alpha E : \alpha \in K^*\}$

    (d) $Z(UT_n(K)) = \{E + e_{1n} : \alpha \in K\}$

29. Show that $Z(\overline{\prod}_{i \in I} G_i) = \overline{\prod}_{i \in I} Z(G_i)$.

For the next five problems K denotes a field.

30. Show that if $|K| > 2$ or $n > 2$ then $GL_n(K)' = SL_n(K)$.

31. Prove that $GL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2) \cong Sym(3)$. Find $GL_2(\mathbb{F}_2)'$.

32. Show that if $|K| > 2$ or $n \geq 3$ then $SL_n(K)' = SL_n(K)$.

33. Find $SL_2(\mathbb{F}_2)'$. Prove that $SL_2(\mathbb{F}_3)' \neq SL_2(\mathbb{F}_3)$.

34. Show that if $|K| > 2$ then $T_n(K)' = UT_n(K)$. If $|K| = 2$ then $T_n(K)' = UT_n(K)$.

35. Let G be a group, $A, B, C \trianglelefteq G$. Then $[AB, C] = [A, C]\,[B, C]$.

36. Let G be as in problem 25. Find G' and show that any element in G' is a commutator.

37. Prove that $\left(\prod_{i \in I} G_i\right)' = \prod_{i \in I} G_i'$. Is it true for $\overline{\prod}$.

38. Decide which of the following properties are preserved under homomorphisms:

    (a) being abelian,

    (b) being nonabelian,

(c) being finite,

(d) being infinite,

(e) being torsion,

(f) being torsion-free.

39. Show that a quotient of a cyclic group is cyclic.

40. Let G be a group and $N \trianglelefteq G$. Prove that G/N is abelian if and only if $G' \leq N$.

41. Find $T_n(K)/UT_n(K)$, $\left\{ \frac{a}{p^n} : a \in \mathbb{Z}, n \in \mathbb{N} \right\} / \mathbb{Z}$.

42. Let G be a group. Define a map Inn from G to $Aut(G)$ sending g to the automorphism $x \mapsto gxg^{-1}$. Show that

    (a) Inn is a homomorphism,

    (b) $Ker(Inn) = Z(G)$,

    (c) $Inn(G) \trianglelefteq Aut(G)$.

43. Show that $End(K) \cong K$ for $K = \mathbf{Z}, \mathbf{Z}_n, \mathbf{Q}$.

44. Show that $End\left(\oplus_n K\right) \cong M_n(K)$ for $K = \mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$.

45. Let $\mu : A \longrightarrow B$ be a group homomorphism. Prove that $(a, b) \mapsto (a, b.\mu(a)) \in Aut(A \times B)$.

46. Show that $Aut(Sym(3)) = Inn(Sym(3))$.

47. Let G be a group. Let $\Phi \subset End(G)$. If a subgroup H is invariant under all $\varphi \in \Phi$ we write $H \leq_\Phi G$. Let $H_i \leq_\Phi G$. Show that

    (a) $\bigcap_{i \in I} H_i \leq_\Phi G$,

    (b) $\left\langle \bigcup_{i \in I} H_i \right\rangle \leq_\Phi G$.

48. Let $H \leq_\Phi G$. If $\Phi = End(G)$ we write $H \leq_e G$, if $\Phi = Aut(G)$ we write $H \leq_a G$. Show that

    (a) $\leq_e, \leq_a$are transitive,

    (b) $\trianglelefteq$ is not transitive in general,

    (c) If $B \trianglelefteq G$ and $A \leq_a B$ then $A \trianglelefteq G$.

49. Show that $Z(G) \leq_a G$ but in general $Z(G) \not\leq_e G$.

50. Show that in $\mathbb{Z}, \mathbb{Z}_n$ and $\mathbb{C}(p^\infty)$ all subgroups are invariant under endomorphisms.

51. Let K be a field. Show that

(a) $UT_n(K) \leq_e T_n(K)$,
(b) $SL_n(K) \leq_e GL_n(K)$.

52. Show that a periodic abelian group is automorphically simple if and only if it is isomorphic to $\oplus_\lambda \mathbb{Z}_p$ for some prime p.

53. Show that an abelian group is automorphically simple if and only if it is isomorphic to $\oplus_\lambda \mathbb{Z}_p$ for some prime p or $\oplus_\lambda \mathbb{Q}$.

54. Define $Hol(G) = G \times_{semi} Aut(G)$. Show that $Hol(K) = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a \in K, b \in K^* \right\}$ for $K = \mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$.

55. Show that the following groups are nonisomorphic: $\mathbb{Z}Wr\mathbb{Z}_2$, $\mathbb{Z}_2 Wr\mathbb{Z}$, $\mathbb{Z}_2 wr\mathbb{Z}$.

56. Show that if $A_0 \leq A$, $B_0 \leq B$ then $A_0 wr B_0 \hookrightarrow AwrB$.

57. Suppose that $A \neq 1$, $1 \neq N \trianglelefteq AwrB$. Show that $N \cap fun(B, A) \neq 1$.

58. Compute $Z(AwrB)$, $Z(AWrb)$.

59. Compute $(\mathbb{Z}wr\mathbb{Z})'$.

60. Show that the operation wr is not associative even in the class of finite groups.

61. Let $\mathbb{R}^*$ act on $\mathbb{R}^2$ by

    (a) $\lambda(x, y) = (\lambda x, \lambda^{-1} y)$,
    (b) $\lambda(x, y) = (\lambda x, y)$,
    (c) $\lambda(x, y) = (\lambda x, \lambda^2 y)$,
    (d) $\lambda(x, y) = (\lambda^2 x, \lambda^2 y)$.

    Find the orbits.

62. Let $Sym(n)$ act on $\mathbb{R}^n$ via permuting coordinates. Find the orbits.

63. Let $Sym(\mathbb{N})$ act on $\wp(\mathbb{N})$ in the natural way. Find the orbits.

64. Show that for each prime p and each integer greater than 2 there is a nonabelian group of order p^n.

65. Let G be a finite nontrivial p-group acting on a finite set. Show that the kernel of the action is nontrivial.

66. Show that $\mathbb{R}^{>0}$ splits in $\mathbb{R}^*$.

67. Decide whether the following groups split or not:

    (a) $PSL_n(K)$ in $GL_n(K)$

(b) $PSL_n(K)$ in $PGL_n(K)$

(c) $\mathbb{Z}$ in $\mathbb{Q}$

(d) $n\mathbb{Z}$ in $\mathbb{Z}$

68. Prove that in a nilpotent group G the set $A_p = \{g \in G : g^{p^n} = 1$ for some $n\}$ is a subgroup of G. Show that if G is torsion then G is the direct sum of A_p's.

69. Show that a divisible torsion nilpotent group is abelian.

70. Let G be a nilpotent group, $H \lhd G$. Show that if G is divisible and torsion then it is central in G.

71. Find $End(\mathbb{Z}/n\mathbb{Z})$, $Aut(\mathbb{Z}/n\mathbb{Z})$ and $Hom(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/m\mathbb{Z})$.

72. Find $End((\mathbb{Z}/n\mathbb{Z})^m)$, $Aut((\mathbb{Z}/n\mathbb{Z})^m)$, $End(\mathbb{Z}/p^n\mathbb{Z})$ and $End(\mathbb{Z}/p^\infty\mathbb{Z})$.

73. Let g be an element of $PSL_n(K)$. Show that the centralizer of g is abelian if and only if $charK = 2$ and $n = 2$. Prove that this never happens for $Alt(n)$ for $n \geq 5$.

74. Let G act transitively on X. Let $x \in X$. Prove that the action of G is equivalent to the action of G on G/G_x by left translation.

75. An abelian group is called reduced if it has no divisible nonzero subgroups. Prove that any abelian group is a direct sum of a divisible and a reduced group.

76. In any torsion free divisible abelian group the intersection of any family of divisible subgroups is divisible. Show that for torsion groups it fails.

77. Prove that for any torsion-free abelian group A there is a torsion free divisible abelian group $\overline{A} \geq A$ such that there is no divisible group B with $\overline{A} \geq B \geq A$. $\overline{A}$ is called a divisible hall of A. Show that all the divisible halls of A are isomorphic over A.

78. Let $G = \overline{\bigoplus}_p \mathbb{Z}_p$. Show that $G/G_{tor} \cong (\mathbb{R}, +)$.

79. Prove that any direct sum of cyclic groups is reduced.

80. Show that $\mathbb{Q}$ and $\mathbb{C}(p^\infty)$ are not direct sums.

81. Prove that if G/H is torsion free then H is a pure subgroup of G

82. Show that cyclic groups has no pure subgroups.

83. Prove that in the group $F(a, b)$ the set $\{a^n b a^n : n \in \mathbb{N}\}$ generates a subgroup freely.

84. Prove that every free group with more than one generator is nonabelian.

85. Prove that
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free subgroup of $Sl_2(\mathbb{Z})$.

86. Find defining relations for $\oplus_n \mathbb{Z}$.

87. Show that $Sym(3) \cong \langle a, b : a^2, b^3, (ab)^2 \rangle$, $\mathbf{Z}_6 \cong \langle a, b : a^2, b^3, a^{-1}b^{-1}ab \rangle$.

88. Prove that if $F(X) \cong F(Y)$ then $|X| = |Y|$.

89. Let KN be the class of all K by N groups. Show that $K(NM) = (KN)M$.

90. Let F be a field. Prove that $T_n(K)$ is solvable.

91. Show that in any group the product of two solvable subgroups is solvable subgroup.

92. Show that in any finite group G there is the greatest normal solvable subgroup S of G. Moreover G/S has no normal abelian subgroups.

93. Prove that any finitely generated torsion solvable group is finite.

94. Prove that for any finite solvable group there is a series of subgroups
$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$
such that $G_i \trianglelefteq G$ and G_{i+1}/G_i is cyclic.

95. Let $A \leq B \leq C$ be groups. Then $B/A \subseteq Z(G/A)$ if and only if $[B, C] \leq A$.

96. Let A be the class of abelian groups. Show that if G is nilpotent of class n then $G \in A^n$.

97. Let K be a ring, $n \geq 2$. Prove that $UT_n(K)$ is nilpotent of class $n - 1$.

98. Show that for any group G if $G/Z(G)$ is nilpotent then G is nilpotent.