Math 211 Algebra

Final

January 16, 2006-01-14

A module is called **simple** if it has no nontrivial proper submodules. A module is called **semisimple** if it is a direct sum of simple modules.

1. Classify all simple $\mathbb{Z}$-modules. Find a $\mathbb{Z}$-module which is not semisimple.
2. Let R be a ring and N and M be two simple R-modules. Show that any R-module homomorphism from N into M is an isomorphism. Dedule that $\mathrm{End}_R(M)$ is a division ring.
3. Show that the following three conditions on a module M are equivalent:
   a) M is a direct sum of simple submodules.
   b) M is a sum of simple submodules.
   c) Every submodule N of M is a direct summand of M, that is, there is a submodule N' such that $M = N \oplus N'$.

   Note: You need Zorn's Lemma.
4. Conclude that submodules and quotients of semisimple modules are semisimple.