MATH 311
GROUP THEORY
Final exam
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1. Prove that the additive group of rational numbers is indecomposable into a direct sum.
2. An equivalence relation $\equiv$ on a group G is called a congruence if it agrees with the group operation, that is, $x \equiv x'$ and $y \equiv y'$ imply $xy \equiv x'y'$. For a subgroup H of G let $x \equiv_H y$ be defined by $xH = yH$. Prove that
   (1) $\equiv_H$ is a congruence of G iff H is a normal subgroup of G,
   (2) any congruence of G is of the form $\equiv_H$, for some subgroup H of G.
3. Prove that every nonabelian group of order 6 is isomorphic to S_3.
4. Let G be an abelian p-group without elements of infinite height. Prove that if G has finitely many elements of order p then G is finite.
5. A group G is called complete if $Z(G) = 1$ and any automorphism of G is inner. Prove that if G is complete then
   (1) $\mathrm{Aut}(G) \simeq G$,
   (2) for any group H, if G is a normal subgroup of H then this extension splits.
6. Let H be a subgroup of index $\leq n$ of a group G. Prove that H contains a normal subgroup of G of index $\leq n!$. (Hint: consider the action of G on the set of right cosets of H by right translations.)
7. Let A and B be groups, $A \neq 1$, and B is infinite. Prove that $Z(A \,\mathrm{wr}\, B) = 1$.
8. Prove that in the free group $F(x, y)$ the set $\{[x^n, y] : \ n = 1, 2, \dots\}$ is a basis of the subgroup generated by the set.
9. Prove that any minimal normal subgroup of a finite solvable group is a direct sum of cyclic subgroups of same prime order.