MATH 311 Basic Group Theory
Prof. Oleg Belegradek
Problem set 10

1. For classes of groups \( \mathcal{K} \) and \( \mathcal{N} \) we denote by \( \mathcal{K}\mathcal{N} \) the class of all \( \mathcal{K} \)-by-\( \mathcal{N} \) groups. Prove that \( (\mathcal{K}\mathcal{N})\mathcal{M} = \mathcal{K}(\mathcal{N}\mathcal{M}) \), for any classes of groups \( \mathcal{K}, \mathcal{N}, \mathcal{M} \).

2. Prove that for any field \( F \) and any \( n \geq 1 \) the group \( T_n(F) \) is \( n \)-step solvable.

3. Prove that for any \( n \) the class \( \mathfrak{X}_n \) is closed under subgroups, cartesian products and homomorphic images. Is the same true for the class of all solvable groups?

4. Prove that solvable-by-solvable group is solvable.

5. Prove that in any group the product of two normal solvable subgroups is a normal solvable subgroup.

6. Prove that in any finite group \( G \) there is the greatest normal solvable subgroup \( R \), the so called solvable radical of \( G \). Show that \( G/R \) has no normal solvable subgroups.

7. Prove that any finitely generated periodic solvable group is finite.

8. Prove that if \( G \) is a finite solvable group then there are subgroups

\[ \{e\} = G_0 < G_1 < \cdots < G_n = G \]

such that \( G_i \triangleleft G_{i+1} \) and \( G_{i+1}/G_i \) is a cyclic group of prime order for any \( i < n \).

9. For a group \( G \) and \( i \leq \omega \) we define inductively the subgroup \( \zeta_i(G) \) as follows:

\[ \zeta_0(G) = G, \quad \zeta_{i+1}(G) = [\zeta_i(G), G]. \]

(a) Show that all subgroups \( \zeta_i(G) \) are normal, and \( \zeta_{i+1}(G) \leq \zeta_i(G) \) for all \( i \).

(b) Show that \( G \) is nilpotent iff \( \zeta_n(G) = \{e\} \) for some \( n \). Moreover, \( G \) is \( n \)-step nilpotent iff \( \zeta_n(G) = \{e\} \) but \( \zeta_{n-1}(G) \neq \{e\} \).

(c) Suppose \( G \) is a nilpotent group and \( \{e\} = G_n \leq G_{n-1} \leq \cdots \leq G_0 = G \) is a central series in \( G \). Prove that \( \zeta_i(G) \leq G_i \) for all \( i \leq n \).