MATH 311 Basic Group Theory
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Problem set 10

1. For classes of groups $\mathcal{K}$ and $\mathcal{N}$ we denote by $\mathcal{K}\mathcal{N}$ the class of all $\mathcal{K}$-by-$\mathcal{N}$ groups. Prove that $(\mathcal{K}\mathcal{N})\mathcal{M} = \mathcal{K}(\mathcal{N}\mathcal{M})$, for any classes of groups $\mathcal{K}$, $\mathcal{N}$, $\mathcal{M}$.
2. Prove that for any field F and any $n \geq 1$ the group $T_n(F)$ is n-step solvable.
3. Prove that for any n the class $\mathfrak{A}^n$ is closed under subgroups, cartesian products and homomorphic images. Is the same true for the class of all solvable groups?
4. Prove that solvable-by-solvable group is solvable.
5. Prove that in any group the product of two normal solvable subgroups is a normal solvable subgroup.
6. Prove that in any finite group G there is the greatest normal solvable subgroup R, the so called solvable radical of G. Show that G/R has no normal solvable subgroups.
7. Prove that any finitely generated periodic solvable group is finite.
8. Prove that if G is a finite solvable group then there are subgroups
$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$
such that $G_i \lhd G_{i+1}$ and G_{i+1}/G_i is a cyclic group of prime order for any $i < n$.
9. For a group G and $i < \omega$ we define inductively the subgroup $\zeta_i(G)$ as follows: $\zeta_0(G) = G$, $\zeta_{i+1}(G) = [\zeta_i(G), G]$.
   (a) Show that all subgroups $\zeta_i(G)$ are normal, and $\zeta_{i+1}(G) \leq \zeta_i(G)$ for all i. The series $\zeta_0(G) \geq \zeta_1(G) \geq \ldots$ is called the *lower central series* of G.
   (b) Show that G is nilpotent iff $\zeta_n(G) = \{e\}$ for some n. Moreover, G is n-step nilpotent iff $\zeta_n(G) = \{e\}$ but $\zeta_{n-1}(G) \neq \{e\}$.
   (c) Suppose G is a nilpotent group and $\{e\} = G_n \leq G_{n-1} \leq \cdots \leq G_0 = G$ is a central series in G. Prove that $\zeta_i(G) \leq G_i$ for all $i \leq n$.