MATH 311 Basic Group Theory
Prof. Oleg Belegradek
Problem set 9

1. Prove that a cyclic group has no pure subgroups.
2. Prove that A is a pure subgroup of an abelian group B iff $A \cap p^n B = p^n A$ for any prime p and any $n \geq 1$.
3. Let B be an abelian group and $A \leq B$. Show that if B/A is torsion-free then A is a pure subgroup of B. Is the converse true?
4. Let p be a prime. Prove that any infinite abelian p-group without elements of infinite height has infinitely many elements of order p.
5. Show that there are no nonzero groups which are free in the class of all abelian periodic group.
6. Classify groups free in the class of all abelian groups of exponent n.
7. Prove that any free group is torsion-free.
8. Show that a non-cyclic free group has trivial center.
9. Prove that the subgroup of the free group with generators x, y generated by the subset $S = \{x^n y x^n : n = 1, 2, \dots\}$ is free, and S is its basis.
10. Let u, v be elements of a free group F. Prove that $uv = vu$ if and only if the subgroup of F generated by u, v is cyclic.
11. Find finite sets of generators and defining relations for the groups
    (a) $\mathbb{Z}^n$,
    (b) $\mathbb{Z}_2 \times \mathbb{Z}_3$,
    (c) S_3,
    (d) $\mathrm{UT}_3(\mathbb{Z})$.