MATH 311 Basic Group Theory
Prof. Oleg Belegradek
Problem set 8

1. Using integral elementary transformations of rows and columns, transform the integral matrix $\begin{pmatrix} 2 & 2 & 3 \\ 4 & -1 & 2 \\ 3 & 2 & 3 \end{pmatrix}$ to an integral matrix of the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, where $a, b, c \geq 0$ and $a \mid b \mid c$.

2. Prove that the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ generate the group $\mathrm{GL}_2(\mathbb{Z})$. Show that the matrix $\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$ belongs to $\mathrm{GL}_2(\mathbb{Z})$ and represent it as a product of the generators and their inverses.

3. Prove that the cartesian product of any family of finite cyclic groups is reduced (that is, has no divisible nonzero subgroups).

4. Let $\bar{G}$ and G be the cartesian and direct products, respectively, of all groups $\mathbb{Z}_p$, where p runs over the primes. Prove that
   (a) G is the torsion part of $\bar{G}$,
   (b) $\bar{G}/G$ is torsion-free,
   (c) $\bar{G}/G$ is divisible,
   (d) $\bar{G}$ is reduced,
   (e) G is not a direct summand in $\bar{G}$,
   (f) $|\bar{G}/G| = 2^{\aleph_0}$,
   (g) $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \bar{G}/G = 2^{\aleph_0}$,
   (h) $\bar{G}/G \simeq \mathbb{R}$.

5. What is the 3-height of the element $\bar{6}$ in the group $\mathbb{Z}_{81}$?

6. Let p be a prime, and $k, m \in \mathbb{Z}$. Prove that
   (a) there is $\phi \in \mathrm{End}(\mathbb{Z}_{p^n})$ such that $\phi(\bar{k}) = \bar{m}$ if and only if $h_p(\bar{k}) \leq h_p(\bar{m})$;
   (b) there is $\phi \in \mathrm{Aut}(\mathbb{Z}_{p^n})$ such that $\phi(\bar{k}) = \bar{m}$ if and only if $h_p(\bar{k}) = h_p(\bar{m})$.

7. Let p be a prime, and κ_n, ν_n cardinals, $n = 1, 2, \ldots$. Suppose $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}^{(\kappa_n)} \simeq \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}^{(\nu_n)}$. Prove that $\kappa_n = \nu_n$ for all n.