MATH 311 Basic Group Theory
Prof. Oleg Belegradek
Problem set 7

1. Prove that any periodic abelian group is embeddable into a periodic divisible abelian group.
2. Prove that any torsion-free abelian group is embeddable into a torsion-free divisible abelian group.
3. Prove that the intersection of any family of divisible subgroups of a torsion-free abelian group is a divisible subgroup. Show that the intersection of two divisible subgroups of an abelian p-group need not be a divisible subgroup.
4. Prove that the sum of any family of divisible subgroups of an abelian group is a divisible subgroup.
5. An abelian group is called *reduced* if it has no nonzero divisible subgroups. Prove that any abelian group can be decomposed into a direct sum of divisible and reduced subgroups.
6. Prove that $\mathbb{Q}/\mathbb{Z} \simeq \oplus_p \mathbb{C}(p^\infty)$.
7. Prove that finitely generated abelian groups G and H are isomorphic if and only if $|p^nG : p^{n+1}G| = |p^nH : p^{n+1}H|$ for any prime p and non-negative integer n.