Math 211

Midterm

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1. Let I and J be two ideals of a ring R. Assume that $I \subseteq J \subseteq R$. Show that the ring $(R/I)/(J/I)$ is isomorphic to the ring R/J.

2. Let I and J be two ideals of a ring R. Let IJ be the ideal generated by the set $\{ij: i \in I, j \in J\}$. Show that $IJ \subseteq I \cap J$. Does the equality always hold? For each $n > 0$ find an example where $I^n = 0$ but $I^{n-1} \neq 0$.

3. Let $f, g \in \mathbb{Z}[X]$. Assume that a prime number p divides fg in $\mathbb{Z}[X]$, i.e., $fg = ph$ for some $h \in \mathbb{Z}[X]$. Show that p divides f or g.

4. Recall that an R-module M is finitely generated if $M = Rm_1 + ... + Rm_n$ for some $m_1, ..., m_n \in M$.

4a. Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

4b. Let f be a monic polynomial in $\mathbb{Z}[X]$. Show that $\mathbb{Z}[X]/\langle f \rangle$ is a finitely generated $\mathbb{Z}$-module.

4c. Let I be a nonzero ideal of $\mathbb{Z}[X]$. Is $\mathbb{Z}[X]/I$ always a finitely generated $\mathbb{Z}$-module?

5. Let I be an ideal of $\mathbb{Z}[X]$. Show that the ideal of $\mathbb{Q}[X]$ generated by I is of the form $\{fg/n : g \in \mathbb{Z}[X], n \in \mathbb{N} \setminus \{0\}\}$ for some fixed f in I. (Hint: Consider the ideal generated by I in $\mathbb{Q}[X]$).

6. Show that a finite commutative ring with no zerodivisors is a field.

7. Let C be the set of functions from $\mathbb{R}$ into $\mathbb{R}$. C is a ring under the addition and multiplication of functions.

7a. Describe the invertible elements of C.

7b. Describe the set of zero divisors of C

7c. Let $a \in \mathbf{R}$ be fixed. Consider $I_a = \{f \in C : f(a) = 0\}$. Show that I_a is a maximal ideal of C. What is C/I_a?

7d. Find an ideal I of C with the property that $I \setminus I_a \neq \emptyset$. Conclude that there is a maximal ideal of C which is different from the ideals of the form I_a.