1. Let $I$ and $J$ be two ideals of a ring $R$. Assume that $I \subseteq J \subseteq R$. Show that the ring $(R/I)/(J/I)$ is isomorphic to the ring $R/J$.

2. Let $I$ and $J$ be two ideals of a ring $R$. Let $IJ$ be the ideal generated by the set \{ij : i \in I, j \in J\}. Show that $IJ \subseteq I \cap J$. Does the equality always hold? For each $n > 0$ find an example where $I^n = 0$ but $I^{n-1} \neq 0$.

3. Let $f, g \in \mathbb{Z}[X]$. Assume that a prime number $p$ divides $fg$ in $\mathbb{Z}[X]$, i.e., $fg = ph$ for some $h \in \mathbb{Z}[X]$. Show that $p$ divides $f$ or $g$.

4. Recall that an $R$-module $M$ is finitely generated if $M = Rm_1 + \ldots + Rm_n$ for some $m_1, \ldots, m_n \in M$.
   
   4a. Show that $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.
   
   4b. Let $f$ be a monic polynomial in $\mathbb{Z}[X]$. Show that $\mathbb{Z}[X]/\langle f \rangle$ is a finitely generated $\mathbb{Z}$-module.
   
   4c. Let $I$ be a nonzero ideal of $\mathbb{Z}[X]$. Is $\mathbb{Z}[X]/I$ always a finitely generated $\mathbb{Z}$-module?

5. Let $I$ be an ideal of $\mathbb{Z}[X]$. Show that the ideal of $\mathbb{Q}[X]$ generated by $I$ is of the form \{fg/n : g \in \mathbb{Z}[X], n \in \mathbb{N} \setminus \{0\}\} for some fixed $f$ in $I$. (Hint: Consider the ideal generated by $I$ in $\mathbb{Q}[X]$).

6. Show that a finite commutative ring with no zero divisors is a field.

7. Let $C$ be the set of functions from $\mathbb{R}$ into $\mathbb{R}$. $C$ is a ring under the addition and multiplication of functions.
   
   7a. Describe the invertible elements of $C$.
   
   7b. Describe the set of zero divisors of $C$.
   
   7c. Let $a \in \mathbb{R}$ be fixed. Consider $I_a = \{f \in C : f(a) = 0\}$. Show that $I_a$ is a maximal ideal of $C$. What is $C/I_a$?
   
   7d. Find an ideal $I$ of $C$ with the property that $I \setminus I_a \neq \emptyset$. Conclude that there is a maximal ideal of $C$ which is different from the ideals of the form $I_a$. 