MATH 311 Basic Group Theory
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Problem set 6

1. Let F be a finite field, $|F| = p^n$. Show that $\mathrm{UT}_n(F)$ is a Sylow p-subgroup of $\mathrm{GL}_n(F)$.
2. Find Sylow subgroups of the groups S_3 and S_4.
3. Find all Sylow 2-sybgroups of the direct sum of countably many copies of S_3, and show that there are non-conjugate Sylow 2-subgroups.
4. Prove that any group of order 50 is not simple.
5. Let P be a Sylow p-subgroup of a finite group G, and a be a p-element of G. Prove that if $P^a = P$ then $a \in P$.
6. Prove that if P is a Sylow p-subgroup of a finite group G then P is a Sylow subgroup of $N(P)$ and is the only Sylow subgroup of $N(P)$.
7. Prove that if P is a Sylow p-subgroup of a finite group G then $N(N(P)) = N(P)$.
8. Let G be a finite p-group. Prove that $N(H) \neq H$, for any proper subgroup H of G.
9. Prove that any subgroup of order p^{n-1} in a group of order p^n is normal.
10. Let G be a finite group and $\phi : G \to H$ an epimorphism. Prove that
    (a) if P is a Sylow p-subgroup of G then $\phi(P)$ is Sylow p-subgroup of H;
    (b) if Q is a Sylow p-subgroup of H then there is a Sylow p-subgroup P of G such that $Q = \phi(P)$.