MATH 311 Basic Group Theory
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Problem set 4

1. Prove that for the rings $K = \mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$ the holomorph of the additive group of K is isomorphic to the group of all matrices of the form
$$\begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix},$$
where $\alpha, \beta \in K$, $\beta \neq 0$.
2. Prove that if the group A is nontrivial then $N \cap \mathrm{fun}(B, A)$ is nontrivial, for any nontrivial normal subgroup N of $A\mathrm{wr}B$.
3. Prove that if A is nontrivial and B is infinite then
   (a) $Z(A\mathrm{wr}B) = 1$,
   (b) $Z(A\mathrm{Wr}B)$ is the subset of $\mathrm{Fun}(B, A)$ consisting of all constant functions from B to $Z(A)$.
4. Let G be a group, A a normal subgroup of G, and $B = G/A$. Prove that G is embeddable into $A\mathrm{Wr}B$.
   Hint. Let $s : B \to G$ such that $s(hA) \in hA$ for any $h \in G$. For $g, h \in G$ put $f_g(hA) = s(ghA)^{-1}\, g\, s(hA)$. Check that $f_g(hA) \in A$ and so $f_g \in \mathrm{Fun}(B, A)$. Check that the map $g \mapsto gA \cdot f_g$ is a monomorphism from G to $A\mathrm{Wr}B$.
5. Show that wr is a non-associative operation on the class of finite groups.
   Hint. Compare the orders of $(A\mathrm{wr}B)\mathrm{wr}C$ and $A\mathrm{wr}(B\mathrm{wr}C)$ for finite groups A, B, C.
6. Prove that any center-by-cyclic group is abelian.
7. Show that a group of order p^2 is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$, for a prime p.
8. Give an example of a nonabelian group of order p^3.
9. Give an example of infinite p-group with trivial center.