MATH 311 Basic Group Theory
Prof. Oleg Belegradek
Problem set 4

1. Prove that $T_n(K)/UT_n(K) \simeq D_n(K)$.
2. Prove that $\mathbb{Z}_p/Z \simeq \mathbb{C}(p^\infty)$.
3. Prove that any group of cardinality $> 2$ has a non-identity automorphism.
4. Prove that $\text{Im}(G) \leq \text{Aut}(G)$.
5. Prove that $\text{Im}(G) \simeq G/Z(G)$.
6. Show that all automorphisms of $S_3$ are inner.
7. Let $G$ and $H$ be groups. Prove that for any homomorphism $\mu : G \to Z(H)$
   the map $(g, h) \mapsto (g, h + \mu(g))$ is an automorphism of $G \times H$.
8. For an abelian group $A$ we denote by $\text{End}(A)$ its endomorphism ring. For a
   ring $R$ let $R_+$ be its additive group. Prove that for the rings $R = \mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$
   (a) $\text{End}(\mathbb{Z}_n) \simeq \mathbb{Z}$,
   (b) $\text{End}(\mathbb{Z}_n^+)$ \simeq $\mathbb{M}_n(R)$,
   (c) $\text{Aut}(\mathbb{Z}_n^+)$ \simeq $\mathbb{GL}_n(R)$,
   (d) Is $\text{End}(\mathbb{C}_n)$ \simeq $\mathbb{C}$?
   (e) Is $\text{End}(\mathbb{R}_+)$ \simeq $\mathbb{R}$?
9. Let $\Phi$ is a set of endomorphisms of a group $G$. Prove that
   (a) the intersection of any set of $\Phi$-invariant subgroups is $\Phi$-invariant,
   (b) the subgroup generated by a $\Phi$-invariant subset is $\Phi$-invariant.
10. Let $H \leq G$. We write $H \leq_a G (H \leq_e G)$ if $H$ is invariant under any
    automorphism (endomorphism) of $G$. Prove that
    (a) the relations $\leq_a$ and $\leq_e$ are transitive;
    (b) the relation $\leq_a$ is not transitive;
    (c) If $A \leq_a B$ and $B \leq G$ then $A \leq G$.
11. Prove that in the groups $\mathbb{Z}, \mathbb{Z}_n, \mathbb{C}(p^\infty)$ all subgroups are invariant under all
    endomorphisms.
12. Let $K$ be a field. Prove that
    (a) $\text{SL}_n(K) \leq_e \text{GL}_n(K)$,
    (b) $\text{UT}_n(K) \leq_e T_n(K)$.
13. Prove that
    (a) the additive group of any vector space has no proper nonzero subgroups
        which are invariant under all endomorphisms;
    (b) if an abelian group has no proper nonzero subgroups which are invariant
        under all endomorphisms then it is the additive group of a vector space
        over a field.
    (c) Which abelian groups are additive groups of vector spaces?