MATH 311 Basic Group Theory
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Problem set 4

1. Prove that $\mathrm{T}_n(K)/\mathrm{UT}_n(K) \simeq D_n(K)$.
2. Prove that $\mathbb{Z}_{(p)}/\mathbb{Z} \simeq \mathbb{C}(p^\infty)$.
3. Prove that any group of cardinality > 2 has a non-identity automorphism.
4. Prove that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.
5. Prove that $\mathrm{Inn}(G) \simeq G/Z(G)$.
6. Show that all automorphisms of S_3 are inner.
7. Let G and H be groups. Prove that for any homomorphism $\mu : G \to Z(H)$ the map $(g, h) \to (g, h + \mu(g))$ is an automorphism of $G \times H$.
8. For an abelian group A we denote by $\mathrm{End}(A)$ its endomorphism ring. For a ring R let R_+ be its additive group. Prove that for the rings $R = \mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$
   (a) $\mathrm{End}(R_+) \simeq R$,
   (b) $\mathrm{End}(R_+^n)) \simeq \mathrm{M}_n(R)$,
   (c) $\mathrm{Aut}(R_+^n)) \simeq \mathrm{GL}_n(R)$.
   (d) Is $\mathrm{End}(\mathbb{C}_+) \simeq \mathbb{C}$?
   (e) Is $\mathrm{End}(\mathbb{R}_+) \simeq \mathbb{R}$?
9. Let Φ is a set of endomorphisms of a group G. Prove that
   (a) the intersection of any set of Φ-invariant subgroups is Φ-invariant,
   (b) the subgroup generated by a Φ-invariant subset is Φ-invariant.
10. Let $H \leq G$. We write $H \leq_a G$ ($H \leq_e G$) if H is invariant under any automorphism (endomorphism) of G. Prove that
    (a) the relations $\leq_a$ and $\leq_e$ are transitive;
    (b) the relation $\trianglelefteq$ is not transitive;
    (c) If $A \leq_a B$ and $B \trianglelefteq G$ then $A \trianglelefteq G$.
11. Prove that in the groups $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{C}(p^\infty)$ all subgroups are invariant under all endomorphisms.
12. Let K be a field. Prove that
    (a) $\mathrm{SL}_n(K) \leq_e \mathrm{GL}_n(K)$,
    (b) $\mathrm{UT}_n^m(K) \leq_e \mathrm{T}_n(K)$.
13. Prove that
    (a) the additive group of any vector space has no proper nonzero subgroups which are invariant under all endomorphisms;
    (b) if an abelian group has no proper nonzero subgroups which are invariant under all endomorphisms then it is the additive group of a vector space over a field.
    (c) Which abelian groups are additive groups of vector spaces?