MATH 311 Basic Group Theory
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Problem set 3

1. Show that if A, B, C are normal subgroups of G then $[AB, C] = [A, C][B, C]$.
2. Let K be a field, $n \geq 2$. Prove that in the group $\mathrm{GL}_n(K)$
   (a) $[t_{ij}(\alpha), t_{jk}(\beta)] = t_{ik}(\alpha\beta)$,
   (b) $[t_{ij}(\alpha), \mathrm{diag}(\beta_1, \dots, \beta_n)] = t_{ij}(\alpha\beta_j\beta_i^{-1} - \alpha)$.
3. Let K be a field, $n \geq 2$.
   (a) Show that if $n \geq 3$ or $|K| \geq 3$ then $\mathrm{GL}_n(K)' = \mathrm{SL}_n(K)$.
   (b) Show that $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2) \simeq S_3$. Find $\mathrm{GL}_2(\mathbb{F}_2)'$.
   (c) Show that if $n \geq 3$ or $|K| \geq 4$ then $\mathrm{SL}_n(K)' = \mathrm{SL}_n(K)$.
   (d) Find $|\mathrm{GL}_2(\mathbb{F}_3)|$ and $|\mathrm{SL}_2(\mathbb{F}_3)|$.
   (e) Show that $\mathrm{SL}_2(\mathbb{F}_3)' \neq \mathrm{SL}_2(\mathbb{F}_3)$.
   (f) Show that if $|K| \geq 3$ then $\mathrm{T}_n(K)' = \mathrm{UT}_n(K)$.
   (g) Show that $\mathrm{T}_n(\mathbb{F}_2) = \mathrm{UT}_n(\mathbb{F}_2)$, and $\mathrm{T}_n(\mathbb{F}_2)' = \mathrm{UT}_n^2(\mathbb{F}_2)$.
   (h) Show that $[\mathrm{UT}_n^r(K), \mathrm{UT}_n^s(K)] = \mathrm{UT}_n^{r+s}(K)$, where $\mathrm{UT}_n^m(K)$ is considered to be trivial for $m \geq n$.