MATH 311 Basic Group Theory
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Problem set 2

1. Prove that the group $\mathbb{Q}$ has no subgroups of finite index.
2. (a) Prove that $|A : A \cap B| \leq |G : B|$ for any subgroups A, B of a group G.
   (b) Prove that if $A \leq B \leq G$ then $|G : A|$ is finite iff $|G : B|$ and $|B : A|$ both are finite. Moreover, if $|G : A|$ is finite then
   $$|G : A| = |G : B| \cdot |B : A|.$$
   (c) Prove that the intersection of finitely many subgroups of finite index is a subgroup of finite index.
3. Prove that two permutations are conjugate in $\mathrm{Sym}(X)$ iff they have the same number of orbits of cardinality n for every $n \in \{1, 2, \ldots, \aleph_0\}$.
4. (a) Prove that any subgroup of index 2 is normal.
   (b) Give an example which shows that for 3 instead of 2 this is not true.
5. Let K be a field.
   (a) Prove that $\mathrm{SL}_n(K) \trianglelefteq \mathrm{GL}_n(K)$.
   (b) Prove that $\mathrm{UT}_n(K) \trianglelefteq \mathrm{T}_n(K)$.
   (c) Prove that $\mathrm{UT}_n^m(K) \trianglelefteq \mathrm{UT}_n(K)$. Here, for $1 \leq m \leq n$, $\mathrm{UT}_n^m(K)$ is defined to be the set of matrices (a_{ij}) in $\mathrm{UT}_n(K)$ such that $a_{ij} = 0$ if $0 < j - i < m$.
6. Let G be the subgroup $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathrm{GL}_2(\mathbb{Q})$. Find in G a subgroup H conjugate with a proper subgroup of H.
7. Let H be a subgroup of finite index in G, and K be the intersection of all subgroups conjugate with H. Prove that K is a normal subgroup of finite index in G.
8. Prove that $Z(A_n) = \{e\}$ for $n \geq 4$, and A_3 is abelian.
9. Let K be a field. Prove that
   (a) $Z(\mathrm{GL}_n(K)) = \{\alpha E : \alpha \in K\}$,
   (b) $Z(\mathrm{SL}_n(K)) = \{\alpha E : \alpha \in K, \alpha^n = 1\}$,
   (c) $Z(\mathrm{T}_n(K)) = \{\alpha E : \alpha \in K\}$, if $|K| \neq 2$,
   (d) $Z(\mathrm{UT}_n(K)) = \{e + \alpha e_{1n} : \alpha \in K\}$.
   (e) $Z(\mathrm{T}_n(K)) = \{e + \alpha e_{1n} : \alpha \in K\}$, if $|K| = 2$.
10. Prove that the centralizer of any finite normal subgroup of G is a subgroup of finite index in G.