1. Prove that the group \( \mathbb{Q} \) has no subgroups of finite index.
2. (a) Prove that \( |A : A \cap B| \leq |G : B| \) for any subgroups \( A, B \) of a group \( G \).
   (b) Prove that if \( A \leq B \leq G \) then \( |G : A| \) is finite iff \( |G : B| \) and \( |B : A| \) both are finite. Moreover, if \( |G : A| \) is finite then
   \[ |G : A| = |G : B| : |B : A|. \]
   (c) Prove that the intersection of finitely many subgroups of finite index is a subgroup of finite index.
3. Prove that two permutations are conjugate in \( \text{Sym}(X) \) iff they have the same number of orbits of cardinality \( n \) for every \( n \in \{1, 2, \ldots, N_0\} \).
4. (a) Prove that any subgroup of index 2 is normal.
   (b) Give an example which shows that for 3 instead of 2 this is not true.
5. Let \( K \) be a field.
   (a) Prove that \( \text{SL}_n(K) \leq \text{GL}_n(K) \).
   (b) Prove that \( \text{UT}_n(K) \leq \text{T}_n(K) \).
   (c) Prove that \( \text{UT}_n^m(K) \leq \text{UT}_n(K) \). Here, for \( 1 \leq m \leq n \), \( \text{UT}_n^m(K) \) is defined to be the set of matrices \( (a_{ij}) \) in \( \text{UT}_n(K) \) such that \( a_{ij} = 0 \) if \( 0 < j - i < m \).
6. Let \( G \) be the subgroup \( \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right\} \) of \( \text{GL}_2(\mathbb{Q}) \). Find in \( G \) a subgroup \( H \) conjugate with a proper subgroup of \( H \).
7. Let \( H \) be a subgroup of finite index in \( G \), and \( K \) be the intersection of all subgroups conjugate with \( H \). Prove that \( K \) is a normal subgroup of finite index in \( G \).
8. Prove that \( Z(A_n) = \{e\} \) for \( n \geq 4 \), and \( A_3 \) is abelian.
9. Let \( K \) be a field. Prove that
   (a) \( Z(\text{GL}_n(K)) = \{ \alpha E : \alpha \in K \} \),
   (b) \( Z(\text{SL}_n(K)) = \{ \alpha E : \alpha \in K, \alpha^n = 1 \} \),
   (c) \( Z(\text{T}_n(K)) = \{ \alpha E : \alpha \in K \} \), if \( |K| \neq 2 \),
   (d) \( Z(\text{UT}_n(K)) = \{ e + \alpha e_{1n} : \alpha \in K \} \),
   (e) \( Z(\text{T}_n(K)) = \{ e + \alpha e_{1n} : \alpha \in K \} \), if \( |K| = 2 \).
10. Prove that the centralizer of any finite normal subgroup of \( G \) is a subgroup of finite index in \( G \).