MATH 311 Basic Group Theory
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Problem set 1

1. Prove that
   (a) $(\mathbb{R}, +) \cong (\mathbb{R}^{>0}, \cdot)$;
   (b) $(\mathbb{Q}, +) \not\cong (\mathbb{Q}^{>0}, \cdot)$.
2. Find the automorphism group of the graph with vertices $\{1,2,3,4,5\}$ and edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{1,4\}$, $\{1,5\}$, $\{2,5\}$, $\{3,5\}$, $\{4,5\}$.
3. Find the group of all isometries of
   (a) the square $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$;
   (b) the set $\mathbb{Z}^2$ on the plane.
4. Prove that
   (a) $\mathbb{Q} = \langle 1/p^n : \ p \text{ is prime}, \ n \in \mathbb{Z}, n > 0 \rangle$;
   (b) $\mathbb{Q}^* = \langle -1, 2, 3, 5, \dots \rangle$;
   (c) $\mathbb{C}(p^\infty) = \langle \varepsilon_{p^n} : \ n \in \mathbb{Z}, n > 0 \rangle$, where $\varepsilon_k = \cos \frac{2\pi}{k} + i \sin \frac{2\pi}{k}$;
   (d) $S_n = \langle (12), (12 \dots n) \rangle$.
5. Let F be a field, n a positive integer, $1 \le i, j \le n$, and $i \ne j$. Let $\alpha \in F$. We call the $n \times n$ matrices of the form $t_{ij}(\alpha) = e + \alpha e_{ij}$ transvections. Let $d(\alpha)$ be the diagonal $n \times n$ matrix (a_{ij}) with $a_{ii} = 1$ for $i < n$ and $a_{nn} = \alpha$. Prove that
   (a) the group $\mathrm{GL}_n(F)$ is generated by all transvections and all matrices of the form $d(\alpha)$ with $\alpha \ne 0$.
   (b) the group $\mathrm{SL}_n(F)$ is generated by all transvections.
   (c) the group $\mathrm{T}_n(F)$ is generated by all transvections $t_{ij}(\alpha)$ with $i < j$ and all diagonal matrices with nonzero elements on the diagonal.
   (d) the group $\mathrm{UT}_n(F)$ is generated by all transvections $t_{ij}(\alpha)$ with $i < j$.
   Hint: show that every matrix can be represented as $t_1 \dots t_r d(\alpha) t_{r+1} \dots t_s$, where the t_i's are transvections.
6. Let elements a and b of a group commute, $|a| = n$, and $|b| = m$.
   (a) Prove that $|ab| = mn$ if $(m,n) = 1$.
   (b) Is it true that always $|ab| = \mathrm{lcm}(a,b)$?
   (c) Prove that there is an element $c\langle a, b\rangle$ with $|c| = \mathrm{lcm}(a,b)$.
7. Prove that
   (a) $\mathbb{C}(nm) \cong \mathbb{C}(n) \times \mathbb{C}(m)$ if $(m,n) = 1$.
   (b) $\mathbb{C}(p^n)$ is indecomposable into direct product, for any prime p and positive integer n.
8. Show that for any $n, m, k \in \{2, 3, 4, \dots\} \cup \{\infty\}$ there are elements σ, τ, ρ in the group of permutations of $\mathbb{N}$ such that such that $|\sigma| = n$, $|\tau| = m$, $|\rho| = k$.
9. (a) Prove that any proper subgroup of $\mathbb{C}(p^\infty)$ is $\mathbb{C}(p^n)$ for some n.
   (b) Prove that if all proper subgroups of an infinite abelian group G are finite then $G \cong \mathbb{C}(p^\infty)$ for some prime p.
10. Prove that any finitely generated periodic abelian group is finite.