MATH 212
BASIC ALGEBRA 2
Midterm exam
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1. Let $n > 1$ and $\alpha_1, \dots, \alpha_n$ be all roots of the polynomial $x^n + x^{n-1} + 1$. Find $\alpha_1^2 + \cdots + \alpha_n^2$.
2. Prove that the field $\mathbb{C}(x)$ of rational functions over complex numbers is not algebraically closed.
3. Let K be a splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$. What is the degree of K over $\mathbb{Q}$? Find a basis of K as a vector space over $\mathbb{Q}$.
4. Decompose $9-3i$ into a product of irreducible elements in the ring of Gaussian numbers $\mathbb{Z}[i]$.
5. Let $\mathbb{H}$ be the division ring of quaternions, and $1, i, j, k$ be its standard basis.
   (a) For $a = 4 - 8i$ and $b = 1 - i + j - k$, find ab^{-1} and $b^{-1}a$ in $\mathbb{H}$.
   (b) Prove that $\mathbb{H} \simeq \mathbb{H}^{\text{op}}$.
6. Prove that $A \otimes_{\mathbb{Z}} \mathbb{Z}_n \simeq A/nA$, for any abelian group A.
7. Let $p(x)$ be a real polynomial which takes only nonnegative values. Prove that there exist real polynomials $f(x)$ and $g(x)$ such that $p = f^2 + g^2$.