Math 211 Final Exam
(Group Theory)
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1. Let $n > 1$ be an integer and let m be a divisor of n.
   a) Show that $\mathbb{Z}/n\mathbb{Z}$ has m elements whose order divides m.
   b) How many elements does $\mathbb{Z}/n\mathbb{Z}$ have whose order is exactly m?
   c) Show that a subgroup of a cyclic group is cyclic.
2. Let p be a prime. Suppose that G has a normal p-subgroup. Show that G has a normal and nontrivial abelian subgroup.
3. Let $p \leq q$ be two primes and G a group of order pq. Show that if $q \not\equiv 1 \bmod p$ then G is abelian.
4. Let $p < q$ be two primes and G a group of order pq^2. Show that if $q \not\equiv \pm 1 \bmod p$ then G is abelian.
5. Let $p < q$ be two primes with $q \equiv 1 \bmod p$.
   a) Show that there are at most p nonisomorphic groups of order pq.
   b) Show that the upper bound p may be attained.
6. Let p and q be two primes and n a natural number. Let G be a group of order $p^n q$.

a) Show that if $q \not\equiv 1 \bmod p$ then G has a nontrivial normal abelian subgroup.
b) Show that there is a sequence $1 = G_0 \leq G_1 \leq ... \leq G_n \leq G_{n+1}$ of normal subgroups such that G_{i+1}/G_i is of prime order.

7. Give the correct mathematical definition of the following "definition": A finite group G is called ***solvable*** if either $G = 1$ or there is a nontrivial normal abelian subgroup A such that G/A is solvable.
8. a) Show that if H_1 and H_2 are two subgroups of finite index of the group G, then $H_1 \cap H_2$ is a subgroup of finite index of G.
   b) Show that if H is a subgroup of finite index of the group G, then H has finitely many conjugates.
   c) Show that if H is a subgroup of finite index of the group G, then there is a normal subgroup N of finite index in G such that $N \subseteq H$.
9. Let $H \leq G$ be a subgroup of finite index, say n. Let $X = G/H = \{xH : x \in G\}$ (the left coset space). For $g \in G$ and $xH \in X$, define $\varphi_g(xH) = gxH$.
   a) Show that φ_g is a bijection of X, so that $\varphi_g \in \text{Sym}(X)$.
   b) For $g \in G$, let $\varphi(g) = \varphi_g \in \text{Sym}(X)$. Show that φ is a group homomorphism from G into $\text{Sym}(X)$.
   c) Show that $\text{Ker}(\varphi) = \cap_{g \in G} H^g \leq H$.
   d) Show that $[G : \text{Ker}(\varphi)]$ divides $n!$
   e) Compare this with #8c.
10. Let G be a finite group and p, the smallest prime that divides $|G|$. Let H be a subgroup of G of index p. Show that $H \lhd G$.