MATH 211
BASIC ALGEBRA II
Final exam
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1. Let a, b, c be the complex roots of the polynomial $x^3 - x - 1$. Find a polynomial over $\mathbb{Q}$ whose roots are a^2, b^2, c^2.
2. Find all quaternions q such that $q^2 + 1 = 0$.
3. Divide $10 + 3i$ by $1 + i$ with remainder in the ring of Gaussian numbers $\mathbb{Z}[i]$. (Remember that the ring $\mathbb{Z}[i]$ is euclidian with the "degree" function $d(z) = |z|^2$.)
4. Prove that $M \otimes R \simeq M$, for any commutative ring R and R-module M. (Here $M \otimes R$ is the tensor product of the R-module M and the R-module R.)
5. Let R be the ring of all rational numbers m/n with odd n.
   (a) Which elements of R are invertible?
   (b) Which elements of R are irreducible?
   (c) Is the ring R factorial?
   (d) Is the ring R a principal ideal domain?
   (e) Is the ring R euclidian?
   (f) Is the polynomial $\frac{1}{3}x^5 + 2x + \frac{2}{5}$ irreducible over R?
6. Prove that for any finite field F of prime characteristic p the map $x \mapsto x^p$ is an automorphism of F. Give an example of a field of prime characteristic p for which the map $x \mapsto x^p$ is not an automorphism.
7. Let F be a finite field. Prove that $F[x]$ contains irreducible polynomials of arbitrarily high degree.