MATH 211
BASIC ALGEBRA 1
Final exam
Prof. Oleg Belegradek

1. Prove that any nonempty subset of a finite group closed under the group operation is a subgroup. Show that for infinite groups it is not true.

2. For which $n$ the group $S_5$ has an element of order $n$?

3. Prove that the groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are isomorphic only if $n = m$.

4. Prove that any group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5. Prove that for finite abelian groups a converse of Lagrange’s theorem holds: if $G$ is a finite abelian group of order $n$ then for any positive divisor $d$ of $n$ there is a subgroup of $G$ of order $d$.

6. For which integers $n$ the quotient ring $\mathbb{Q}[x]/(x^2 - n)\mathbb{Q}[x]$ is a field?

7. Let $R$ be a nonzero commutative ring without ideals except for $R$ and 0. Prove that either $R$ is a field, or $R$ is of prime order and $xy = 0$ for all $x, y \in R$.

8. Let $p$ be a prime, and $\alpha \in \mathbb{C}$, $\alpha \neq 1$, $\alpha^p = 1$. Find the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

9. Prove that the only automorphism of the field of real numbers is the identity mapping.