BASIC ALGEBRA 1

Midterm exam, 2000

1. Prove that for any subgroup N of $\mathbb{Q}$ the quotient group $\mathbb{Q}/N$ is not cyclic.
2. Let $\mathbf{U}$ be the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$. Prove that
   (a) $\mathbb{R}/\mathbb{Z} \simeq \mathbf{U}$,
   (b) $\mathbf{U}/\mathbb{C}(p^n) \simeq \mathbf{U}$.
3. Let N be the set of all real $n \times n$ matrices with a positive determinant. Prove that N is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$. Find $\mathrm{GL}_n(\mathbb{R})/N$.
4. Let F be a field, $a \in F$, and I the ideal in $F[x]$ generated by $x-a$. Prove that $F[x]/I \simeq F$.
5. Let I the ideal in $\mathbb{R}[x]$ generated by $x^2 + x + 1$. Prove that
$$\mathbb{R}[x]/I \simeq \mathbb{C}.$$
6. Let F be a field, and I the ideal in $F[x]$ generated by x^n, where $n > 0$. Let $p(x)$ be a polynomial over F. Prove that $p + I$ is invertible in $F[x]/I$ iff $p(0) \neq 0$.
7. Let F be a field, and $p(x)$, $q(x)$ be co-prime polynomials over F. Prove that
$$F[x]/(pq)F[x] \simeq F[x]/p(x)F[x] \oplus F[x]/q(x)F[x].$$
8. Are the quotient rings $\mathbb{Z}/(x^2 - 2)\mathbb{Z}[x]$ and $\mathbb{Z}/(x^2 - 3)\mathbb{Z}[x]$ isomorphic?