MATH 211 Basic Algebra

Problem Set 8

1. Find the ring of fractions of $\mathbb{Z}_6$ over the multiplicative subset $\{1, 2, 4\}$.
2. Let $C[0,1]$ be the ring of continuous real functions on $[0,1]$. For $0 \leq c \leq 1$ let I_c be the the set of all f in $C[0,1]$ such that $f(c) = 0$. Prove that
   (a) I_c is a maximal ideal of $C[0,1]$,
   (b) any maximal ideal of $C[0,1]$ is of the form I_c for some c.
3. Find all maximal ideals of the ring $\mathbb{Z}_n$.
4. Show that a skew field has no proper non-zero one-side (that is, left or right) ideals.
5. Let R be a associative ring with nonzero multiplication (that is, there are $x, y \in R$ such that $xy \neq 0$). Suppose R has no proper nonzero one-side ideals. Prove that R is a skew field. (Hint: first prove that R has no zero divisors, then prove that R has a unit, and use these facts to prove that R is a skew field.)
6. Let R be an associative ring with unit and without zero divizors. Suppose for any descending sequence of left ideals $I_1 \supseteq I_2 \supseteq \ldots$ there is n_0 such that $I_n = I_{n_0}$ for $n \geq n_0$. Prove that R is a skew field.
7. Prove that for any field F the ring $M_n(F)$ has no proper nonzero ideals.
8. Prove that the ideals of the ring $M_n(\mathbb{Z})$ are exactly the subsets $M_n(k\mathbb{Z})$, where $k \in \mathbb{Z}$.
9. Find the greatest common divisor of the rational polynomials
$$x^4 + x^3 - 3x^2 - 4x - 1 \quad \text{and} \quad x^3 + x^2 - x - 1$$
   and its linear expression in terms of them.
10. Find the greatest common divisor of the polynomials over the two-element field $\mathbb{F}_2$
$$x^5 + x^4 + 1 \quad \text{and} \quad x^4 + x^2 + 1$$
   and its linear expression in terms of them.
11. Let $f(x), g(x), h(x)$ be polynomials over a field F. Prove that f is coprime with g and h then f is coprime with gh.
12. Calculate $\varphi(2002)$, $\varphi(12000)$, where φ is the Euler function.
13. Find the remainder of 3^{578} modulo 308.
14. Prove that the quotient-rings $\mathbb{Q}[x]/(x^2-2)\mathbb{Q}[x]$ and $\mathbb{Q}[x]/(x^2-3)\mathbb{Q}[x]$ are fields. Are they isomorphic?
15. Is the quotient ring $\mathbb{Q}[x]/(x^2-1)\mathbb{Q}[x]$ a field?