MATH 211 Basic Algebra

Problem Set 8

1. Find the ring of fractions of \( \mathbb{Z}_0 \) over the multiplicative subset \{1, 2, 4\}.

2. Let \( C[0, 1] \) be the ring of continuous real functions on \([0, 1]\). For \( 0 \leq c \leq 1 \) let \( I_c \) be the set of all \( f \) in \( C[0, 1] \) such that \( f(c) = 0 \). Prove that
   (a) \( I_c \) is a maximal ideal of \( C[0, 1] \);
   (b) any maximal ideal of \( C[0, 1] \) is of the form \( I_c \) for some \( c \).

3. Find all maximal ideals of the ring \( \mathbb{Z}_n \).

4. Show that a skew field has no proper non-zero one-side (that is, left or right) ideals.

5. Let \( R \) be a associative ring with non-zero multiplication (that is, there are \( x, y \in R \) such that \( xy \neq 0 \)). Suppose \( R \) has no proper nonzero one-side ideals. Prove that \( R \) is a skew field. (Hint: first prove that \( R \) has no zero divisors, then prove that \( R \) has a unit, and use these facts to prove that \( R \) is a skew field.)

6. Let \( R \) be an associative ring with unit and without zero divisors. Suppose for any descending sequence of left ideals \( I_1 \supseteq I_2 \supseteq \ldots \) there is \( n_0 \) such that \( I_n = I_{n_0} \) for \( n \geq n_0 \). Prove that \( R \) is a skew field.

7. Prove that for any field \( F \) the ring \( M_n(F) \) has no proper nonzero ideals.

8. Prove that the ideals of the ring \( M_n(\mathbb{Z}) \) are exactly the subsets \( M_n(k\mathbb{Z}) \), where \( k \in \mathbb{Z} \).

9. Find the greatest common divisor of the rational polynomials
   \[ x^4 + x^3 - 3x^2 - 4x - 1 \quad \text{and} \quad x^3 + x^2 - x - 1 \]
   and its linear expression in terms of them.

10. Find the greatest common divisor of the polynomials over the two-element field \( F_2 \)
    \[ x^5 + x^4 + 1 \quad \text{and} \quad x^4 + x^2 + 1 \]
    and its linear expression in terms of them.

11. Let \( f(x), g(x), h(x) \) be polynomials over a field \( F \). Prove that \( f \) is coprime with \( g \) and \( h \) then \( f \) is coprime with \( gh \).

12. Calculate \( \varphi(2002), \varphi(12000) \), where \( \varphi \) is the Euler function.

13. Find the remainder of \( 3^{578} \) modulo 308.

14. Prove that the quotient-rings \( \mathbb{Q}[x]/(x^2 - 2)\mathbb{Q}[x] \) and \( \mathbb{Q}[x]/(x^2 - 3)\mathbb{Q}[x] \) are fields. Are they isomorphic?

15. Is the quotient ring \( \mathbb{Q}[x]/(x^2 - 1)\mathbb{Q}[x] \) a field?