MATH 211 Basic Algebra

Problem Set 7

1. Which of the following sets are $\mathbb{R}$-subalgebras of the $\mathbb{R}$-algebra $M_n(\mathbb{R})$?
   (a) the symmetric matrices,
   (b) the matrices A such that ${}^tA$ is the inverse of A,
   (c) the upper triangular matrices,
   (d) the upper unitriangular matrices,
   (e) the upper triangular matrices with zero diagonal,
   (f) the matrice with zero last row.
2. Does the set of real polynomials $\mathbb{R}[x]$ form a ring with respect to the ordinary addition and the composition operation?
3. Let R be the set of all subsets of a set X. For $a, b \in R$ put $a \cdot b = a \cap b$ and $a + b = (a \setminus b) \cup (b \setminus a)$. Is $(R, +, \cdot)$ a ring?
4. Prove that for any abelian group A the set $\mathrm{End}(A)$ of all endomorphisms of A forms an associative ring with unit, if for $f, g \in \mathrm{End}(A)$ and $a \in A$ we define $(f+g)(a) = f(a) + g(a)$ and $(fg)(a) = f(g(a))$.
5. Prove that the ring $\mathrm{End}(R, +)$ is isomorphic to R for the rings $R = \mathbb{Z}, Z_n, \mathbb{Q}$.
6. For quaternions $q = 2 - i + 2j - 3k$ and $r = 1 + 2i - j + 2k$ find qp, pq, qp^{-1}, $p^{-1}q$, pq^{-1}, $q^{-1}p$.
7. Find the subgroup of $\mathbb{H}^*$ generated by i, j. What is the order of this subgroup?
8. Prove that the real matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ form a subalgebra of the $\mathbb{R}$-algebra $M_2(\mathbb{R})$ isomorphic to the $\mathbb{R}$-algebra of complex numbers $\mathbb{C}$.
9. Prove that the complex matrices of the form $\begin{pmatrix} z & u \\ -\bar{u} & \bar{z} \end{pmatrix}$ form a subalgebra of the $\mathbb{R}$-algebra $M_2(\mathbb{C})$ isomorphic to the $\mathbb{R}$-algebra of quaternions $\mathbb{H}$.
10. Prove that $\mathbb{R}[x]/(x-2)\mathbb{R}[x] \simeq \mathbb{R}$.
11. Is $\mathbb{R}[x]/(x^2-2)\mathbb{R}[x] \simeq \mathbb{R}$?
12. Prove that the set S of all numbers $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$, forms a subfield of $\mathbb{R}$. Show that $\mathbb{Q}[x]/(x^2-2)\mathbb{Q}[x] \simeq S$.
13. Prove that the ring of all upper triangular real $n \times n$ matrices is indecomposable into a direct product.