MATH 211 Basic Algebra

Problem Set 6

1. Prove that no subgroup of $\mathbb{Q}$ is decomposable into a direct sum of nonzero subgroups.
2. Prove that that the group S_3 is indecomposable into a direct product of nontrivial subgroups.
3. Prove that for any prime p the groups $\mathbb{C}(p^\infty)$ and $\sum_{n=1}^{\infty} \mathbb{Z}_{p^n}$ are not isomorphic.
4. Prove that
$$\mathrm{Tor}(\prod_{n=1}^{\infty} \mathbb{Z}_{p_n}) = \sum_{n=1}^{\infty} \mathbb{Z}_{p_n},$$
where p_n is the nth prime.
5. Let $G = \{\pm 2^n : n \in \mathbb{Z}\}$.
   (a) Show that G is a subgroup of $\mathbb{Q}^*$.
   (b) Prove that G is decomposable into a direct product of two nontrivial subgroups.
   (c) Prove that there is only one decomposition of G into a direct product of two nontrivial subgroups.
6. Prove that the group $\mathbb{Q}^*$ is decomposable into a direct sum of a cyclic group of order 2 and countably many infinite cyclic subgroups.
7. Prove that the additive groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$ are isomorphic iff $n = m$.
8. Decompose into a direct sum of nonzero sugroups the groups $\mathbb{Z}_6$, $\mathbb{Z}_{12}$, $\mathbb{Z}_{60}$.
9. Are the following groups isomorphic?
   (a) $\mathbb{Z}_6 \times \mathbb{Z}_{36}$ and $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$,
   (b) $\mathbb{Z}_6 \times \mathbb{Z}_{36}$ and $\mathbb{Z}_9 \times \mathbb{Z}_{24}$,
   (c) $\mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{60} \times \mathbb{Z}_{10}$.
10. Find all, up to isomorphism, abelian groups of orders 8, 12, 16, 24, 36, 48.
11. How many elements of orders 2,4 and 5 are there in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$?
12. Find all direct decompositions of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$, and $\mathbb{Z}_2 \times \mathbb{Z}_4$.
13. Prove that, for finite abelian groups A, B, C, if $A \times C \simeq B \times C$ then $A \simeq B$.
14. Prove that if the order of a finite abelian group A is divided by n then A has a subgroup of order n.
15. Let A and B be finite abelian groups. Prove that A is isomorphic to B if and only if A and B has the same number of elements of order n, for any natural number n.
16. Let A and B be finite abelian groups. Prove that A is isomorphic to B if and only if $|p^n A : p^{n+1} A| = |p^n B : p^{n+1} B|$, for any prime p and any integer $n \geq 0$.