Algebra

(Math 221 for Math)
Final Exam

January 1999
Ali Nesin & Özlem Beyarslan

1. How many elements does $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ have?

2. Show that $< 5, 5X^6 + X^2 + 2 >$ is a maximal ideal of the ring of polynomials $\mathbb{Z}[X]$.

3. Find a maximal ideal of the ring of continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

4. Let R be a commutative ring with 1. An element e is called **idempotent** if $e^2 = e$.

4a. Show that if e and f are idempotents, then so are $1 - e$, ef and $e + f - ef$. Show that a ring without zerodivisors has only 2 idempotents.

4b. Defining $\otimes$ and $\oplus$ on the set of $I(R)$ of the idempotents as,

$$e \otimes f = ef$$

and

$$e \oplus f = e + f - 2ef$$

show that $I(R)$ becomes almost a ring.

4c. On $I(R)$ define the relation $\leq$ by

$$e \leq f \Leftrightarrow ef = e.$$

Show that $\leq$ is an order on $I(R)$.

4d. Show that for all $e, f \in I(R)$, $ef \leq e$.

4e. An element $e \in I(R) \setminus \{0\}$ is called **atomic** if it is a minimal nonzero element of $I(R)$. Show that if e and f are two different atomic idempotents, then $ef = 0$.

4f. Show that for $e \in I(R)$, Re is a ring with identity (but not necessarily a subring of R).

4g. Show that for $e \in I(R)$, $R \approx Re \oplus R(1 - e)$ (as rings).

4h. Find the idempotents of the ring $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Find its atomic idempotents.

4i. Find the idempotents of the ring $\mathbb{R}[X]/ <X^2 - X>$. Find its atomic idempotents and its decomposition as in 4g.

4j. Let us call a ring **atomic** if every descending chain of idempotents is stationary. Show that every ring with DCC on the ideals (such a ring is called **Artinian**) is atomic.

4k. Find a nonatomic ring.

5. Let G be a group of order mn where $(m, n) = 1$. Assume that G has two normal subgroups K and H of order m and n respectively. Show that $G \approx H \times K$.

6. Conclude from 5 and the Sylow Theorems that a group of order 15 is cyclic.